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Event-triggered design for optimal output consensus of high-order multi-agent systems

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ABSTRACT

In this paper, we study the optimal output consensus problem for networked linear multi-agent systems. Existing distributed algorithms usually rely on continuous communication among neighbouring agents or continuous update of each agent's local controller. To save communication and computation resources, we apply event-triggered technique to this optimal output consensus problem. We first constructively develop an event-triggered algorithm with a set of applicable parameters relying on event-triggered communication and control and show its effectiveness in solving the problem by rigorous proofs. Then we extend it to the case where the triggering conditions only have to be checked in some sampling time instants. Two simulation examples are given to illustrate the efficacy of our designs.

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Optimal consensus; multi-agent system; event-triggered design; Zeno behaviour

1. Introduction

Multi-agent consensus problem has been actively investigated for many years due to its wide applications in machine learning, smart grid, and sensor networks. Particularly, developing distributed algorithms to drive the given multi-agent system to reach an optimal consensus have received a large amount of attention recently (Yang et al., 2019). This problem is a natural extension of existing distributed optimisation results for single integrators to non-trivial dynamic agents and has been found in many practical scenarios especially when some engineering multi-agent systems are involved (S. Li et al., 2014; Zhang et al., 2017). Since the physical plants are often described by continuous-time models, many efforts have been made to develop continuous-time algorithms to achieve the optimal consensus goal for different classes of high-order multi-agent systems, e.g. (An & Yang, 2022; L. Li et al., 2022; T. Liu et al., 2022; Tang et al., 2019; X. Wang et al., 2020; Xie & Lin, 2017).

In the digital implementation of such continuous-time algorithms, the controller update and information sharing are only allowed to happen at some discrete time instants instead of the whole time interval $[0, \infty)$. A typical treatment is to develop their time-scheduled periodic counterparts, i.e. the controller and shared information are updated periodically according to some fixed periods. Nevertheless, such algorithms may be inefficient as the periods for control execution and information sharing are usually determined by some worst-case analysis. To deal with this issue, event-triggered techniques have been extensively studied for multi-agent consensus coordination problems to save the computation and communication burden (see an excellent survey paper (Nowzari et al., 2019)). In such event-triggered designs, the agents do not have to update its controller or share its own information with neighbouring agents periodically. These actions are taken only if they are necessary according to some extra real-time aperiodic mechanisms.

Our goal is to apply event-triggered technique to the optimal consensus problems and develop effective event-triggered optimal consensus algorithms for a group of high-order multi-agent systems. In fact, there have been several results available in the literature discussing event-triggered optimal consensus/distributed optimisation designs along this technical line. For example, Kia et al. (2015) considered the optimal consensus/distributed optimisation problem for single integrators under discrete (event-triggered) communication conditions. Similar designs were delivered in S. Liu et al. (2016), Deng et al. (2016), Kajiyama et al. (2018), C. Liu et al. (2019), and M. Li et al. (2020) to meet different optimisation requirements. Since these results were exclusively derived for single-integrator multi-agent systems, some recent interesting attempts were further made for second-order or linear agents in Yi et al. (2018), D. Wang et al. (2018), Q. Wang et al. (2019), Z. Li et al. (2020), Yu and Chen (2021), and Xian et al. (2024). However, most aforementioned event-triggered designs are only limited to the communication aspect. That is, the actuator/controller of each agent is assumed to be capable of instantly reacting to the received information from neighbours and continuously updating. When the agents are physical ones, their actuators and the communication sensors may have different clocks. Even they have the same clock, the controllers may not have to update as frequent as the communication sensors. Thus, this continuous controller update requirement might be too demanding and consume a larger number of computation resources and energy than they necessarily need. As a result, it is natural for us to further take the controller update issue into consideration and seek more efficient optimal output consensus algorithms.

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Motivated by the aforementioned observations, we aim at optimal consensus controllers for a general class of linear multi-agent systems in the event-triggered control and event-triggered communication setting. That is, the controller update and information sharing do not have to be implemented in a continuous or synchronously discrete fashion. Thus, the expected optimal consensus controller is inherently asynchronous and of a multi-rate nature. This will inevitably make the parameter choice and performance analysis (particularly, the Zeno-free property) of our algorithms much more challenging than existing optimal consensus designs with merely event-triggered control or event-triggered communication.

The contribution of this paper can be summarised as follows:

- Compared with existing event-triggered optimal consensus results, we formulate and solve the problem for general linear multi-agent systems. The agent dynamics can strictly cover the integrator-type agents investigated in Yi et al. (2018), D. Wang et al. (2018), Z. Li et al. (2020), and Yu and Chen (2021) as special cases. Moreover, the communication topology is allowed to be directed and weight-balanced other than undirected graphs in Yi et al. (2018), D. Wang et al. (2018), Z. Li et al. (2020), and Yu and Chen (2021).
- Depending upon whether continuous monitoring of triggering conditions is required or not, we constructively propose two different kinds of novel event-triggered controllers for heterogeneous multi-agent systems. Contrast to most event-trigged optimal consensus algorithms, the proposed algorithms take both event-triggered controller update and information sharing issues into consideration. This substantially saves the communication and computation resources and enlarges the potential application of such optimal consensus algorithms for high-order multi-agent systems.

The rest of this paper is organised as follows: We first state our problem in Section 2. Then we present the main results of this paper in Section 3. Finally, two examples are given in Section 4 with some concluding remarks in Section 5.

Notations: Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathbb{R}^{n\times m}$ be the set of all $n \times m$ real matrices. $\mathbf{1}_n$ (or $\mathbf{0}_n$) denotes an n-dimensional all-one (or all-zero) column vector and $\mathbf{1}_{n\times m}$ (or $\mathbf{0}_{n\times m}$) all-one (or all-zero) matrix. $\mathrm{col}\,(a_1, \ldots, a_n) = [a_1^\top, \ldots, a_n^\top]^\top$ for column vectors a_i $(i = 1, \ldots, n)$. Let $M_1 = \frac{1}{\sqrt{N}}\mathbf{1}_N$ and M_2 be the matrices satisfying $M_2^\top M_1 = \mathbf{0}_{N-1}$, $M_2^\top M_2 = \mathbb{I}_{N-1}$, and $M_2 M_2^\top = \mathbb{I}_N - M_1 M_1^\top$. For a vector x (or matrix A), $\|x\|$ (or $\|A\|$) denotes its Euclidean (or spectral) norm.

2. Problem statement

We consider the distributed coordination problem for linear multi-agent systems

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i u_i \\ y_i &= C_i x_i, \quad i = 1, \ldots, N \end{aligned} \tag{1}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^m$, and $y_i \in \mathbb{R}^q$ are the state, input, and output of agent i. The system matrices A_i, B_i, C_i are constant with compatible dimensions.

Suppose that each agent $i \in \mathcal{N} \triangleq \{1, \ldots, N\}$ has a private cost function $f_i : \mathbb{R}^q \to \mathbb{R}$ with bounded Hessians in the following sense.

Assumption 2.1: For all $i \in \mathcal{N}$ and $s \in \mathbb{R}^q$, f_i is twice continuously differentiable and satisfies that $\underline{h}\mathbb{I}_q \le \nabla^2 f_i(s) \le \bar{h}\mathbb{I}_q$ for some constants $\bar{h} \ge \underline{h} > 0$.

Define the global cost function for these agents by $f(s) = \sum_{i=1}^N f_i(s)$ for any given $s \in \mathbb{R}^q$. Under Assumption 2.1, function f has a unique minimal solution $y^* \in \mathbb{R}^q$. We are going to develop effective rules for these multi-agent systems to drive their outputs to reach an optimal consensus in the sense that $\lim_{t\to\infty} \|y_i(t) - y_j(t)\| = 0$ and $\lim_{t\to\infty} y_i(t) = y^*$ for all $i \in \mathcal{N}$.

Since the local cost function f_i is private to agent i, we are interested in distributed optimal consensus algorithms for (1). For this purpose, we use a weighted directed graph $\mathcal{G} = \{\mathcal{N}, \mathcal{E}, \mathcal{A}\}$ to describe the information sharing relationship among these agents with node set $\mathcal{N}$, edge set $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$, and weighted adjacency matrix $\mathcal{A} \in \mathbb{R}^{N\times N}$. The entry a_{ij} at the ith row and jth column of $\mathcal{A}$ is strictly positive if there exists a directed edge from node j to node i and is 0 otherwise. Here a directed edge $(i, j) \in \mathcal{E}$ means agent j can get access to the information of agent i. Let $\mathcal{N}_i = \{j \in \mathcal{N} \mid (j, i) \in \mathcal{E}\}$ be the set of all immediate neighbours of agent i that it can communicate with. More graph notations can be found in Mesbahi and Egerstedt (2010).

Assumption 2.2: Digraph $\mathcal{G}$ is strongly connected and weight-balanced.

Let $\mathrm{Sym}\,(L) = \frac{L+L^\top}{2}$ with L the Laplacian of digraph $\mathcal{G}$. It can be verified that $\mathrm{Sym}\,(L)$ is positive semidefinite with all eigenvalues being real under Assumption 2.2. Then we can further order its eigenvalues as $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_N$.

This optimal consensus problem has been partially studied in the name of distributed optimisation/optimal coordination for several kinds of high-order multi-agent systems via continuous algorithms (An & Yang, 2022; Z. Li et al., 2020; Tang et al., 2019; Xie & Lin, 2017). However, these continuous optimal consensus algorithms require each agent communicating with its neighbours and updating its own controller in a continuous manner, which may consume larger amounts of communication and computation resources than they necessarily need. In this paper, we are interested in event-triggered optimal consensus designs for multi-agent systems (1).

To be specific, we assume that each agent can broadcast its current information to its neighbours through the underlying communication network described by $\mathcal{G}$ at some discrete instants $\{\tilde{t}_k^i\}$ with $0 = \tilde{t}_0^i < \tilde{t}_1^i < \ldots$ for $i \in \mathcal{N}$ and $k \in \mathcal{Z}^+ = \{0, 1, \ldots\}$. Meanwhile, they decide whether and when to update its local controller u_i to ensure the expected optimal output consensus. Suppose that the controller update time instants are $\{t_k^i\}$ with $0 = t_0^i < t_1^i < \ldots$ for $i \in \mathcal{N}$ and $k \in \mathcal{Z}^+$. Then our problem for the multi-agent systems (1) boils down to develop optimal consensus algorithms of the following form:

$$\begin{aligned} u_i(t) &= h_i^{\mathrm{C}}(x_i(t_k^i),\, \xi_i(t_k^i),\, \tilde{\xi}_j(t_k^i),\, j \in \mathcal{N}_i) \\ \dot{\xi}_i &= g_i^{\mathrm{C}}(x_i(t_k^i),\, \xi_i(t_k^i),\, \tilde{\xi}_j(t_k^i),\, j \in \mathcal{N}_i), \quad t \in [t_k^i,\, t_{k+1}^i) \end{aligned} \tag{2}$$

where h_i^{C} and g_i^{C} are some continuous functions to be specified and ξ_i is the internal state of the compensator at agent i's controller. Moreover, $\tilde{\xi}_j$ represents the broadcast version of the internal state ξ_j by agent j defined as $\tilde{\xi}_j(t) = \xi_j(\tilde{t}_k^j)$ for $t \in [\tilde{t}_k^j, \tilde{t}_{k+1}^j)$.

Compared with the aforementioned continuous algorithms, the information sharing communication and controller update in rule (2) happen only at some discrete time instants instead of over the time interval $[0, \infty)$. This treatment will definitely save many communication and computation resources. In fact, several interesting attempts have been reported in the literature along this technical line (Deng et al., 2016; Kajiyama et al., 2018; Kia et al., 2015; M. Li et al., 2020; C. Liu et al., 2019; S. Liu et al., 2016; D. Wang et al., 2018). While these works were mainly focused on the communication aspect and enforce that $t_i^k = \tilde{t}_i^k$, we remove this limitation and expect more efficient optimal consensus algorithms by further considering event-triggered control and event-triggered communication simultaneously.

3. Main result

In this section, we solve the optimal output consensus problem for multi-agent systems (1) via event-triggered designs.

Before the main results, an extra assumption is made to ensure the tractability of our formulated optimal consensus problem.

Assumption 3.1: For all $i \in \mathcal{N}$, the triplet (C_i, A_i, B_i) is minimal and

$$\operatorname{rank}\begin{bmatrix} A_i & B_i \\ C_i & \mathbf{0} \end{bmatrix} = n_i + q$$

Remark 3.1: Assumption 3.1 specifies the class of high-order agent dynamics that we are interested in. It includes the integrators or minimum-phase dynamics discussed in Xie and Lin (2017), Tang et al. (2019), An and Yang (2022), and Z. Li et al. (2020) as special cases. Moreover, under this assumption, the so-called regulator equations $\mathbf{0} = A_iX_i + B_iU_i$, $\mathbb{I}_q = C_iX_i$ are ensured to have a unique solution pair X_i and U_i with proper dimensions (Huang, 2004).

We first choose a matrix K_{i1} such that $\hat{A}_i \triangleq A_i + B_iK_{i1}$ is Hurwitz and let $K_{i2} \triangleq U_i - K_{i1}X_i$. Denote by λ_{P_i} the maximal eigenvalue of the positive definite matrix P_i such that $\hat{A}_i^\top P_i + P_i\hat{A}_i = -2\mathbb{I}_{n_i}$. According to Theorem 1.7 in Huang (2004), the optimal consensus problem can be trivially solved by a full-information rule $\tilde{u}_i^{\mathrm{o}}(t) = K_{i1}x_i(t) + K_{i2}y^*$ if y^* is globally known by each agent.

Motivated by this observation, we propose the following controller for agent i with both event-triggered control and event-triggered communication:

$$\begin{aligned} u_i(t) &= K_{i1}x_i(t_k^i) + K_{i2}z_i(t_k^i) \\ \dot{z}_i &= -\alpha\nabla f_i(z_i) - \beta\sum_{j=1}^N a_{ij}(\tilde{z}_i - \tilde{z}_j) - \sum_{j=1}^N a_{ij}(\tilde{v}_i - \tilde{v}_j) \\ \dot{v}_i &= \alpha\beta\sum_{j=1}^N a_{ij}(\tilde{z}_i - \tilde{z}_j), \quad t \in [t_k^i,\, t_{k+1}^i) \end{aligned} \tag{3}$$

where $\tilde{z}_i(t)$ and $\tilde{v}_i(t)$ are the broadcast version of $z_i(t)$ and $v_i(t)$ by agent i, and α, β are parameters to be specified later. This controller is distributed and has multiple rates corresponding to the discrete updates for both control and communication.

Let $\tilde{u}_i(t) = K_{i1}x_i(t) + K_{i2}z_i(t)$, $\bar{u}_i(t) = u_i(t) - \tilde{u}_i(t)$, $\bar{z}_i(t) = z_i(t) - \tilde{z}_i(t)$, and $\bar{v}_i(t) = v_i(t) - \tilde{v}_i(t)$. The composite system of (1) and (3) can be written as follows:

$$\begin{aligned} \dot{x}_i &= \hat{A}_ix_i + B_iK_{i2}z_i + B_i\bar{u}_i \\ \dot{z}_i &= -\alpha\nabla f_i(z_i) - \beta\sum_{j=1}^N a_{ij}(\tilde{z}_i - \tilde{z}_j) - \sum_{j=1}^N a_{ij}(\tilde{v}_i - \tilde{v}_j) \\ \dot{v}_i &= \alpha\beta\sum_{j=1}^N a_{ij}(\tilde{z}_i - \tilde{z}_j), \quad t \in [t_k^i,\, t_{k+1}^i), \quad k \in \mathbb{Z}^+ \end{aligned} \tag{4}$$

We propose the following event-triggered mechanisms to determine the triggering time instants for information sharing communication and controller update:

$$\tilde{t}_{k+1}^i = \inf\left\{t > \tilde{t}_k^i \mid \|\mathrm{col}\,(\bar{z}_i(t), \bar{v}_i(t))\| \geq \tilde{c}_0 + \tilde{c}_1\,\mathrm{e}^{-\tilde{\gamma}\,t}\right\} \tag{5}$$

$$t_{k+1}^i = \inf\left\{t > t_k^i \mid \|\bar{u}_i(t)\| \geq c_0 + c_1\,\mathrm{e}^{-\gamma\,t}\right\} \tag{6}$$

where $\tilde{c}_0, \tilde{c}_1$, $\tilde{\gamma}$ and c_0, c_1, γ are nonnegative constants to be specified later. Such event-triggering functions have been widely used in the literature (Guinaldo et al., 2011; Nowzari et al., 2019; Seyboth et al., 2013).

Here is the first main result of this paper.

Theorem 3.1: *Suppose that Assumptions 2.1–3.1 hold. Let* $\alpha \geq \max\left\{1, \frac{2\eta}{\min\{\underline{h}, \lambda_2\}}, \frac{6\bar{h}^2}{\underline{h}\lambda_2}\right\}$, $\beta \geq \max\left\{1, \frac{7\alpha^2\lambda_N^2}{\lambda_2^2}\right\}$ *for some given constant* $\eta > 0$. *Set* $\tilde{c}_0 \geq 0$, $\tilde{c}_1 \geq 0$, $\tilde{c}_0 + \tilde{c}_1 > 0$ *with* $0 < \tilde{\gamma} < \min\{1, \frac{\eta}{2}\}$ *for rule (5) and* $c_0 \geq \tilde{c}_0$, $c_1 \geq 0$, $c_0 + c_1 > 0$ *with* $0 < \gamma < \min\left\{1, \frac{1}{2\lambda_{P_i}}, \tilde{\gamma}\right\}$ *for rule (6). Then, along the trajectory of the composite system* (1) *and* (3) *with triggering rules* (5) *and* (6), *the following assertions hold:*

(a) *There exists a constant* $\tau_{\min} > 0$ *such that* $t_{k+1}^i - t_k^i > \tau_{\min}$ *and* $\tilde{t}_{k+1}^i - \tilde{t}_k^i > \tau_{\min}$ *for all* $i \in \mathcal{N}$ *and* $k \in \mathbb{Z}^+$.
(b) $y_i(t)$ *converges into a ball centred at* y^* *with its radius proportional to* $c_0 + \tilde{c}_0 \geq 0$ *for all* $i \in \mathcal{N}$.
(c) *When* $c_0 = \tilde{c}_0 = 0$, y_i *exponentially converges to the exact optimal solution* y^* *as* t *goes to* ∞ *for all* $i \in \mathcal{N}$.

Before the proof of this theorem, we introduce a key lemma summarising the convergence performance of $z_i(t)$ and $v_i(t)$ along the trajectory of system (4).

Lemma 3.1: *Suppose that Assumptions* 2.1–3.1 *hold and the parameters are chosen according to Theorem* 3.1. *Then, along the* *trajectory of system (4), $z_i(t)$ exponentially converges into a ball centred at y^* with its radius proportional to $\tilde{c}_0 \geq 0$. Moreover, there exists a constant $\tilde{\tau}_{\min} > 0$ such that $\tilde{t}^i_{k+1} - \tilde{t}^i_k > \tilde{\tau}_{\min}$ for all $i \in \mathcal{N}$ and $k \in \mathbb{Z}^+$.*

Proof: To prove this lemma, we first obtain the following compact form:

$$\begin{aligned} \dot{z} &= -\alpha \nabla \hat{f}(z) - \beta(L \otimes \mathbb{I}_q)\tilde{z} - (L \otimes \mathbb{I}_q)\tilde{v} \\ \dot{v} &= \alpha\beta(L \otimes \mathbb{I}_q)\tilde{z} \end{aligned} \tag{7}$$

where $\hat{f}(z) \triangleq \sum_{i=1}^N f_i(z_i)$ with $z = \operatorname{col}(z_1, \ldots, z_N)$, $v = \operatorname{col}(v_1, \ldots, v_N)$, $\tilde{z} = \operatorname{col}(\tilde{z}_1, \ldots, \tilde{z}_N)$, and $\tilde{v} = \operatorname{col}(\tilde{v}_1, \ldots, \tilde{v}_N)$. It can be verified that function $\hat{f}$ is $\underline{h}$-strongly convex while its gradient $\nabla \hat{f}(z)$ is $\bar{h}$-Lipschitz.

Viewing $\tilde{z} - z$ and $\tilde{v} - v$ as perturbations of system (7), we suppose $\operatorname{col}(z^*, v^*)$ to be an equilibrium of the unperturbed system. One can practically verify that $z^* = \mathbf{1}_N \otimes y^*$ under the lemma conditions. Then we perform the coordinate transformation: $\hat{z}_1 = (M_1^\top \otimes \mathbb{I}_q)(z - z^*)$, $\hat{z}_2 = (M_2^\top \otimes \mathbb{I}_q)(z - z^*)$, $\hat{v}_1 = (M_1^\top \otimes \mathbb{I}_q)(v - v^*)$, and $\hat{v}_2 = (M_2^\top \otimes \mathbb{I}_q)[(v + \alpha z) - (v^* + \alpha z^*)]$. It follows that $\dot{\hat{v}}_1 = 0$

$$\begin{aligned} \dot{\hat{z}}_1 &= -\alpha(M_1^\top \otimes \mathbb{I}_q)\mathbf{\Pi} \\ \dot{\hat{z}}_2 &= -\alpha(M_2^\top \otimes \mathbb{I}_q)\mathbf{\Pi} - \beta(M_L \otimes \mathbb{I}_q)\hat{z}_2 \\ &\quad + \alpha(M_L \otimes \mathbb{I}_q)\hat{z}_2 - (M_L \otimes \mathbb{I}_q)\hat{v}_2 + \Delta_1 \\ \dot{\hat{v}}_2 &= -\alpha(M_L \otimes \mathbb{I}_q)\hat{v}_2 + \alpha^2(M_L \otimes \mathbb{I}_q)\hat{z}_2 \\ &\quad - \alpha^2(M_2^\top \otimes \mathbb{I}_q)\mathbf{\Pi} + \alpha\Delta_2 \end{aligned} \tag{8}$$

where $\mathbf{\Pi} \triangleq \nabla \hat{f}(z) - \nabla \hat{f}(z^\star)$, $M_L = M_2^\top L M_2$, $\Delta_1 \triangleq (\beta - \alpha)(M_L M_2^\top \otimes \mathbb{I}_q)\tilde{z} + (M_L M_2^\top \otimes \mathbb{I}_q)\tilde{v}$, and $\Delta_2 \triangleq (M_L M_2^\top \mathbb{I}_q)\tilde{v} - \alpha(M_L M_2^\top \otimes \mathbb{I}_q)\tilde{z}$.

Let $\hat{z} = \operatorname{col}(\hat{z}_1, \hat{z}_2)$ and $W_0(\hat{z}, \hat{v}_2) = \frac{1}{2}\hat{z}^\top \hat{z} + \frac{1}{\alpha^3}\hat{v}_2^\top \hat{v}_2$ with α chosen in the theorem.

We take the time derivative of W_0 along the trajectory of (8) and have

$$\begin{aligned} \dot{W}_0 \leq & -\alpha\underline{h}\|\hat{z}\|^2 - \beta\lambda_2\|\hat{z}_2\|^2 + \alpha\lambda_N\|\hat{z}_2\|^2 + \lambda_N\|\hat{z}_2\|\|\hat{v}_2\| \\ & - \frac{2\lambda_2}{\alpha^2}\|\hat{v}_2\|^2 + \frac{2\lambda_N}{\alpha}\|\hat{z}_2\|\|\hat{v}_2\| \\ & + \frac{2\bar{h}}{\alpha}\|\hat{v}_2\|\|\hat{z}\| + \hat{z}_2^\top\Delta_1 + \frac{2}{\alpha^2}\hat{v}_2^\top\Delta_2 \\ \leq & -\left(\alpha\underline{h} - \frac{3\bar{h}^2}{\lambda_2}\right)\|\hat{z}\|^2 - \frac{\lambda_2}{\alpha^2}\|\hat{v}_2\|^2 + \hat{z}_2^\top\Delta_1 + \frac{2}{\alpha^2}\hat{v}_2^\top\Delta_2 \\ & - \left(\beta\lambda_2 - \alpha\lambda_N - \frac{3\alpha^2\lambda_N^2}{\lambda_2} - \frac{3\lambda_N^2}{\lambda_2}\right)\|\hat{z}_2\|^2 \\ \leq & -\eta W_0 + \frac{\|\Delta_1\|^2}{2\eta} + \frac{\|\Delta_2\|^2}{\alpha\eta} \end{aligned}$$

where we have used Young's inequality to handle the cross terms. Recalling the expressions of Δ_1 and Δ_2, we have

$$\begin{aligned} \dot{W}_0 &\leq -\eta W_0 + \tilde{k}_e\|\operatorname{col}(\tilde{z}, \tilde{v})\|^2 \\ &\leq -\eta W_0 + N\tilde{k}_e(\tilde{c}_0 + \tilde{c}_1\, \mathrm{e}^{-\tilde{\gamma} t})^2 \end{aligned}$$

for some constant $\tilde{k}_e > 0$. Solving this differential inequality or recalling Lemma 9.8 in Khalil (2002), one can conclude that $W_0(t)$ is uniformly ultimately bounded with an ultimate bound $\tilde{\kappa}\tilde{c}_0^2$ for some constant $\tilde{\kappa} > 0$. Recalling the definition of W_0, one can directly conclude that $z_i - y^*$ converges to a ball centred at 0 with its radius proportional to $\tilde{c}_0$. Particularly, when $\tilde{c}_0 = 0$, we have

$$\begin{aligned} W_0(t) &\leq W_0(0)\,\mathrm{e}^{-\eta t} + N\tilde{k}_e\tilde{c}_1^2\int_0^t \mathrm{e}^{-\eta(t-\tau)}\,\mathrm{e}^{-2\tilde{\gamma}\tau}\,\mathrm{d}\tau \\ &\leq k_W[\mathrm{e}^{-2\tilde{\gamma} t} + \mathrm{e}^{-\eta t}] \end{aligned} \tag{9}$$

with $k_W \triangleq W_0(0) + \frac{N\tilde{k}_e\tilde{c}_1^2}{\eta - 2\tilde{\gamma}}$. According to the definition of W_0, this implies the exponential convergence of $z_i(t)$ towards y^* as t tends to ∞.

Next, we seek some $\tilde{\tau}_{\min}$ such that $\tilde{t}^i_{k+1} - \tilde{t}^i_k > \tilde{\tau}_{\min}$. For this purpose, let us consider the evolution of $\operatorname{col}(\bar{z}_i(t), \bar{v}_i(t))$. Since the right-hand side of system (8) is Lipschitz with respect to $\operatorname{col}(\hat{z}(t), \hat{v}_2(t))$ and the two terms Δ_1 and Δ_2 are Lipschitz with respect to $\operatorname{col}(\bar{z}, \bar{v})$, we have

$$\begin{aligned} \|\operatorname{col}(\dot{\bar{z}}_i(t), \dot{\bar{v}}_i(t))\| &\leq \|\operatorname{col}(\dot{z}(t), \dot{v}(t)\| = \|\operatorname{col}(\dot{\hat{z}}(t), \dot{\hat{v}}_2(t)\| \\ &\leq \tilde{h}_M\|\operatorname{col}(\hat{z}(t), \hat{v}_2(t))\| \\ &\quad + \tilde{h}_\Delta\|\operatorname{col}(\bar{z}(t), \bar{v}(t))\| \\ &\leq \sqrt{2\alpha^3}\tilde{h}_M\sqrt{W_0(t)} + \tilde{h}_\Delta\|\operatorname{col}(\bar{z}(t), \bar{v}(t))\| \end{aligned} \tag{10}$$

for some constants $\tilde{h}_M$, $\tilde{h}_\Delta > 0$.

From here, we consider two different cases (i.e. $\tilde{c}_0 \neq 0$ and $\tilde{c}_0 = 0$) to complete the proof. When $\tilde{c}_0 \neq 0$, we can further obtain that $\|\operatorname{col}(\dot{\bar{z}}_i(t), \dot{\bar{v}}_i(t))\| \leq \tilde{h}_M\sqrt{4\alpha^3 k_W} + \tilde{h}_\Delta\sqrt{N}(\tilde{c}_0 + \tilde{c}_1) \triangleq \tilde{k}_u$. Hence, $\|\operatorname{col}(\bar{z}_i(t), \bar{v}_i(t))\| \leq \tilde{k}_u(t - t^*)$. Then the next event will not be triggered before $t^* + \tilde{c}_0/\tilde{k}_u$. When $\tilde{c}_0 = 0$, we use the bound of $W_0(t)$ and conclude that $\|\operatorname{col}(\dot{\bar{z}}_i(t), \dot{\bar{v}}_i(t))\| \leq \tilde{\tilde{c}}_1\left(\mathrm{e}^{-\frac{\eta t}{2}} + \mathrm{e}^{-\tilde{\gamma} t}\right) = 2\tilde{\tilde{c}}_1\,\mathrm{e}^{-\tilde{\gamma} t}$ for positive $\tilde{\tilde{c}}_1 > 0$. Consider the equation $2\tilde{\tilde{c}}_1 s - \tilde{c}_1\,\mathrm{e}^{-\tilde{\gamma} s} = 0$. It has a unique solution $\tilde{\tau}_1 > 0$ by the intermediate value theorem. Moreover, since the left-hand side of this equation is monotone, the following inequality holds for any $t^* < t < t^* + \tilde{\tau}_1$:

$$\|\operatorname{col}(\bar{z}_i(t), \bar{v}_i(t))\| \leq 2\tilde{\tilde{c}}_1\,\mathrm{e}^{-\tilde{\gamma} t^*}(t - t^*) < \hat{c}_1\,\mathrm{e}^{-\tilde{\gamma} t}$$

That is, the next event will not be triggered before $t^* + \tilde{\tau}_1$. In this way, we set $\tilde{\tau}_{\min} = \min\{\tilde{c}_0/\tilde{k}_u, \tilde{\tau}_1\}$ for both cases. The proof is thus complete. ■

With help of Lemma 3.1, we go back to system (4) and give the proof of Theorem 3.1.

Proof of Theorem 3.1: We first show the latter two items and then identify some $\tau_{\min}$ for item (a) to complete the proof.

Letting $\bar{x}_i = x_i - X_i z_i$ gives $\dot{\bar{x}}_i = \hat{A}_i\bar{x}_i + B_i\bar{u}_i - X_i\dot{z}_i$. We set $V_i(\bar{x}_i) = \bar{x}_i^\top P_i\bar{x}_i$. Its time derivative along the trajectory of $\bar{x}_i$ satisfies

$$\dot{V}_i = 2\bar{x}_i^\top P_i[\hat{A}_i\bar{x}_i + B_i\bar{u}_i - X_i\dot{z}_i]$$

$$\leq -\|\bar{x}_i\|^2 + 2\|P_iB_i\|^2\|\bar{u}_i\|^2 + 2\|P_iX_i\|^2|\dot{z}_i|^2$$

Then we recall that $\|\dot{z}_i\|^2 = \|\dot{\tilde{z}}_i\|^2$ and utilise the Lipschitzness of the right-hand side of system (8). It follows that

$$\begin{aligned}\|\dot{z}_i\|^2 &\leq 2\tilde{h}_M^2\|\mathrm{col}\,(\hat{z}(t), \hat{v}_2(t))\|^2 + 2\tilde{h}_\Delta^2\|\mathrm{col}\,(\bar{z}, \bar{v})\|^2 \\ &\leq 4\tilde{h}_M^2\alpha^3 W_0 + 2\tilde{h}_\Delta^2 N(\tilde{c}_0 + \tilde{c}_1\,\mathrm{e}^{-\tilde{\gamma}t})^2\end{aligned}$$

By the inequality (9), we further have

$$\begin{aligned}\|\dot{z}_i\|^2 &\leq 8\tilde{h}_M^2\alpha^3 k_W\,\mathrm{e}^{-2\tilde{\gamma}t} + 2\tilde{h}_\Delta^2 N(\tilde{c}_0 + \tilde{c}_1\,\mathrm{e}^{-\tilde{\gamma}t})^2 \\ &\leq (8\tilde{h}_M^2\alpha^3 k_W + 4\tilde{h}_\Delta^2 N\tilde{c}_1^2)\,\mathrm{e}^{-2\tilde{\gamma}t} + 4\tilde{h}_\Delta^2 N\tilde{c}_0^2\end{aligned} \tag{11}$$

This together with the triggering mechanism (6) implies that

$$\dot{V}_i \leq -\frac{1}{\lambda_{P_i}}V_i + k_\gamma\,\mathrm{e}^{-2\gamma t} + k_c(c_0 + \tilde{c}_0)^2 \tag{12}$$

for $k_\gamma \triangleq 16\max_i\{\|P_iX_i\|^2(\tilde{h}_M^2\alpha^3 k_W + \tilde{h}_\Delta^2 N\tilde{c}_1^2) + \|P_iB_i\|^2 c_1^2\}$ and $k_c \triangleq 8\max_i\{\|P_iB_i\|^2 + \|P_iX_i\|^2\tilde{h}_\Delta^2 N\}$. According to Lemma 9.8 in Khalil (2002), $V_i(t)$ is uniformly ultimately bounded with an ultimate bound proportional to $(c_0 + \tilde{c}_0)^2$. Note that $y_i - y^* = C_i\bar{x}_i + z_i - y^*$. By the definition of V_i and Lemma 3.1, we conclude that $y_i - y^*$ converges to a ball centred at 0 with its radius proportional to $c_0 + \tilde{c}_0$. This is exactly the statement in item (b).

Particularly, when $c_0 = \tilde{c}_0 = 0$, we solve the inequality (12) and have

$$\begin{aligned}V_i(t) &\leq V_i(0)\,\mathrm{e}^{-\frac{t}{\lambda_{P_i}}} + \int_0^t \mathrm{e}^{-\frac{t-\tau}{\lambda_{P_i}}} k_\gamma\,\mathrm{e}^{-2\gamma\tau}\,\mathrm{d}\,\tau \\ &\leq k_{V_i}\left[\mathrm{e}^{-\frac{t}{\lambda_{P_i}}} + \mathrm{e}^{-2\gamma t}\right]\end{aligned} \tag{13}$$

with $k_{V_i} \triangleq V_i(0) + \frac{2\gamma\lambda_{P_i}k_\gamma}{1-2\lambda_{P_i}\gamma}$. By the triangle inequality, this together with Lemma 3.1 implies the exponential convergence of y_i towards y^* as t tends to ∞, i.e. item (c).

Next, we prove item (a). Since we have found a constant $\tilde{\tau}_{\min}$ such that $\tilde{t}_i^{k+1} - \tilde{t}_i^k > \tilde{\tau}_{\min}$, it suffices to confirm a similar property for the controller update time instants. For this purpose, we check the evolution of $\|\bar{u}_i(t)\|$. Suppose that agent i is triggered to update its controller at $t^* \geq 0$. Before the next controller update event, the derivative of $\bar{u}_i(t)$ satisfies that

$$\dot{\bar{u}}_i = -\dot{\tilde{u}}_i = -K_{i1}(\hat{A}_i\bar{x}_i + B_i\bar{u}_i) - K_{i2}\dot{z}_i \tag{14}$$

From (11), there must be some $k_z > 0$ such that $\|\dot{z}_i\| \leq k_z(\tilde{c}_0 + \tilde{c}_1\,\mathrm{e}^{-\tilde{\gamma}t})$. Recalling that $\|\bar{u}_i(t)\| \leq c_0 + c_1\,\mathrm{e}^{-\gamma t}$ before the next triggering time, we further have

$$\begin{aligned}\|\dot{\bar{u}}_i\| &\leq \|K_{i1}\hat{A}_i\bar{x}_i\| + \|K_{i1}B_i\|c_0 + k_z\|K_{i2}\|\tilde{c}_0 \\ &\quad + \|K_{i1}B_i\|c_1\,\mathrm{e}^{-\gamma t} + k_z\|K_{i2}\|\tilde{c}_1\,\mathrm{e}^{-\tilde{\gamma}t} \\ &\leq \|K_{i1}\hat{A}_i\bar{x}_i\| + (\|K_{i1}B_i\|c_1 + k_z\|K_{i2}\|\tilde{c}_1)\,\mathrm{e}^{-\gamma t} \\ &\quad + (\|K_{i1}B_i\| + k_z\|K_{i2}\|)(c_0 + \tilde{c}_0)\end{aligned}$$

Again, we consider two different cases depending upon whether the constant c_0 is 0 or not to determine a lower bound for controller update inter-event intervals.

When $c_0 \neq 0$, from the ultimate boundedness of $\bar{x}_i(t)$, there exists some $k_u > 0$ such that $\|\dot{\bar{u}}_i\| \leq k_u$ for all $i \in \mathcal{N}$. Hence, $\|\bar{u}_i(t)\| \leq \int_{t^*}^t \|\dot{\bar{u}}_i(t)\|\,\mathrm{d}\,t \leq k_u(t - t^*)$. Note that the next event will not be triggered before $\|\bar{u}_i(t)\|$ reaches c_0. Thus, the inter-event intervals for controller updates must be bounded from the below by $\tau_0 = c_0/k_u$.

When $c_0 = 0$, we have $\tilde{c}_0 = 0$ by parameter choices. Recalling the inequality (13), we conclude that $\|\dot{\bar{u}}_i(t)\| \leq \hat{c}_1(\mathrm{e}^{-\frac{t}{2\lambda_{P_i}}} + \mathrm{e}^{-\gamma t}) < 2\hat{c}_1\,\mathrm{e}^{-\gamma t}$ for some constant $\hat{c}_1 > 0$. Since $\bar{u}_i(t^*) = 0$, it follows then $\|\bar{u}_i(t)\| \leq 2\hat{c}_1\,\mathrm{e}^{-\gamma t^*}(t - t^*)$ for $t > t^*$. By similar arguments as in the proof of Lemma 3.1, we consider the equation $2\hat{c}_1 s - c_1\,\mathrm{e}^{-\gamma s} = 0$ and denote its unique solution by $\tau_1 > 0$. Thus, for any $t^* \leq t < t^* + \tau_1$, $\|\bar{u}_i(t)\| < 2\hat{c}_1\,\mathrm{e}^{-\gamma t^*}\tau_1 = c_1\,\mathrm{e}^{-\gamma(t^*+\tau_1)} < c_1\,\mathrm{e}^{-\gamma t}$. This means the next event for rule (6) is not triggered before $t^* + \tau_1$, or the inter-event interval is lower bounded by τ_1.

Overall, we can set $\tau_{\min} = \min\{\tau_0, \tau_1, \tilde{\tau}_{\min}\}$ with $\tilde{\tau}_{\min}$ determined in Lemma 3.1 for both cases. This confirms the statement in item (a) and thus completes the proof. ■

Remark 3.2: The proposed controller (3) consists of two modules, i.e. an event-triggered optimal signal generator to reproduce the expected optimal consensus point and an event-triggered tracking controller for each agent to approach the generated estimate of the optimal solution. This design is adapted from the embedded design scheme in Tang et al. (2019) to the event-triggered setting. Through this structure, we can construct event-triggered mechanisms for information sharing communication and controller update in a separate way.

To implement the event-triggered algorithm (3) with (5) and (6), we have to monitor the triggering conditions in a continuous manner. This incurs additional computation overhead. Next, we provide its periodic checking extension to avoid this issue.

Let $T_k = k\tau$ with $\tau > 0$ a sampling time to be specified later. Then, for $t \in [T_k, T_{k+1})$, we modify the information sharing and controller update rules as follows:

$$\mathrm{col}\,(\tilde{z}_i(t), \tilde{v}_i(t)) = \begin{cases}\mathrm{col}\,(z_i(T_k), v_i(T_k)), & \text{when } \mathcal{C}_1^i(T_k) \geq 0 \\ \mathrm{col}\,(\tilde{z}_i(T_k), \tilde{v}_i(T_k)), & \text{when } \mathcal{C}_1^i(T_k) < 0\end{cases}$$

$$u_i(t) = \begin{cases}\tilde{u}_i(T_k), & \text{when } \mathcal{C}_2^i(T_k) \geq 0 \\ u_i(T_k), & \text{when } \mathcal{C}_2^i(T_k) < 0\end{cases} \tag{15}$$

where $\mathcal{C}_1^i(t) \triangleq \|\mathrm{col}\,(\bar{z}_i(t), \bar{v}_i(t))\| - \tilde{c}_0 - \tilde{c}_1\,\mathrm{e}^{-\tilde{\gamma}t}$ and $\mathcal{C}_2^i(t) \triangleq \|\bar{u}_i(t)\| - c_0 - c_1\,\mathrm{e}^{-\gamma t}$. Apparently, the information sharing or controller update only happens when the corresponding condition is triggered at the periodic sampling time. Thus, no continuous monitoring is required any more in this algorithm.

Since Zeno behaviours are naturally excluded under rule (15), we only have to focus on the convergence issue. Here is the second main result of this paper.

Theorem 3.2: *Suppose that Assumptions* 2.1–3.1 *hold. Choose the same parameters in Theorem 3.1 and let* $0 < \tau < \min\left\{\frac{1}{\tilde{\gamma}}, \frac{1}{\gamma}\right\}$. *Then, along the trajectory of the composite system* (1) *and* (3) *with rule* (15), $y_i(t)$ *converges into a ball centred at* y^* *with its* *radius proportional to $c_0 + \tilde{c}_0 \geq 0$ for all $i \in \mathcal{N}$. Moreover, when $c_0 = \tilde{c}_0 = 0$, $y_i(t)$ exponentially converges to the exact optimal solution y^* as t goes to ∞ for all $i \in \mathcal{N}$.*

Proof: Performing the same coordinate transformations in (4) and (7), we can put the closed-loop system (1) and (3) with rule (15) into the following form:

$$\begin{aligned}\dot{\bar{x}}_i &= \hat{A}_i\bar{x}_i + B_i\bar{u}_i - X_i\dot{z}_i\\ \dot{\hat{z}}_1 &= -\alpha(M_1^\top \otimes \mathbb{I}_q)\boldsymbol{\Pi}\\ \dot{\hat{z}}_2 &= -\alpha(M_2^\top \otimes \mathbb{I}_q)\boldsymbol{\Pi} - \beta(M_L \otimes \mathbb{I}_q)\hat{z}_2\\ &\quad + \alpha(M_L \otimes \mathbb{I}_q)\hat{z}_2 - (M_L \otimes \mathbb{I}_q)\hat{v}_2 + \Delta_1\\ \dot{\hat{v}}_2 &= -\alpha(M_L \otimes \mathbb{I}_q)\hat{v}_2 + \alpha^2(M_L \otimes \mathbb{I}_q)\hat{z}_2\\ &\quad - \alpha^2(M_2^\top \otimes \mathbb{I}_q)\boldsymbol{\Pi} + \alpha\Delta_2\end{aligned}$$

with $\dot{\hat{v}}_1 = 0$. Following a similar procedure as in the proof of Theorem 3.1, we first consider the evolution of $\hat{z}_1(t)$, $\hat{z}_2(t)$, and $\hat{v}_2(t)$, and then go back the whole system.

First, we use the same function $W_0(t)$ defined in the proof of Lemma 3.1 and have

$$\dot{W}_0 \leq -\eta W_0 + \tilde{k}_e\|\mathrm{col}\,(\bar{z}(t),\ \bar{v}(t))\|^2 \tag{16}$$

Consider its evolution over any given time interval $[T_k, T_{k+1})$. From here, it suffices for us to consider the case when $\frac{\eta}{2}W_0(t) \leq \tilde{k}_e\|\mathrm{col}\,(\bar{z}(t),\ \bar{v}(t))\|^2$ for any $t \in [T_k,\ T_{k+1})$. Otherwise, we must have $\dot{W}_0(t) \leq -\frac{\eta}{2}W_0(t)$ at the time when $\frac{\eta}{2}W_0(t) > \tilde{k}_e\|\mathrm{col}\,(\bar{z}(t),\ \bar{v}(t))\|^2$. This implies that $W_0(t)$ is decreasing at these time instants, which does not deteriorate the convergence of $W_0(t)$.

Let $\theta(t) \triangleq \|\mathrm{col}\,(\bar{z}(t),\ \bar{v}(t))\|$. By the inequality (10), we have

$$\dot{\theta} \leq \|\mathrm{col}\,(\dot{\bar{z}}(t),\ \dot{\bar{v}}(t))\| \leq \left\{2\sqrt{\alpha^3\tilde{k}_e/\eta}\tilde{h}_M + \tilde{h}_\Delta\right\}\sqrt{N}\theta \tag{17}$$

Since no matter whether the controller update event is triggered or not at T_k, we always have $\theta(T_k) \leq \tilde{c}_0 + \tilde{c}_1\,\mathrm{e}^{-\tilde{\gamma}T_k}$. Then we solve the inequality (17) and have

$$\begin{aligned}\theta(t) &\leq \mathrm{e}^{\left\{2\sqrt{\alpha^3\tilde{k}_e/\eta}\tilde{h}_M+\tilde{h}_\Delta\right\}\sqrt{N}\tau}[\tilde{c}_0 + \tilde{c}_1\,\mathrm{e}^{-\tilde{\gamma}T_k}]\\ &\leq \mathrm{e}^{\left\{2\sqrt{\alpha^3\tilde{k}_e/\eta}\tilde{h}_M+\tilde{h}_\Delta\right\}\sqrt{N}\tau}[\tilde{c}_0 + \tilde{c}_1\,\mathrm{e}^{-\tilde{\gamma}t}]\end{aligned}$$

for all $t \in [T_k,\ T_{k+1})$. Note that the above inequality holds for any time interval.

With this relationship, we go back to the inequality (16). Recalling the definition of $W_0(t)$, we resort to Lemma 9.8 in Khalil (2002) and conclude that $z_i(t)$ exponentially converges into a ball centred at y^* with its radius proportional to $\tilde{c}_0 \geq 0$.

Next, we consider the tracking part. It is known from the proof of Theorem 3.1 that

$$\dot{V}_i \leq -\frac{1}{\lambda_{P_i}}V_i + 2\|P_iB_i\|^2\|\bar{u}_i\|^2 + \bar{k}_c(\tilde{c}_0 + \tilde{c}_1\,\mathrm{e}^{-\tilde{\gamma}t})^2$$

for some positive constant $\bar{k}_c$. Again, without loss of generality, we only have to consider the case when $\frac{1}{2\lambda_{P_i}}V_i \leq 2\|P_iB_i\|^2\|\bar{u}_i\|^2$ for any $t \in [T_k,\ T_{k+1})$. In this case, we let $\tilde{\theta}_i(t) \triangleq \|\bar{u}_i(t)\|$. By the Lipschitzness of related terms, we can determine some positive constants k'_u and k'_c such that

$$\dot{\tilde{\theta}}_i(t)\| \leq \|\dot{\bar{u}}_i(t)\| \leq k'_u\|\bar{u}_i(t)\| + k'_c(\tilde{c}_0 + \tilde{c}_1\,\mathrm{e}^{-\tilde{\gamma}t})$$

Again, we solve this inequality and use the event-triggering condition. It follows then

$$\begin{aligned}\|\bar{u}_i(t)\| &\leq \mathrm{e}^{k'_u\tau}\|\bar{u}_i(T_k)\| + k'_c\tau\,\mathrm{e}^{-k'_u\tau}(\tilde{c}_0 + \tilde{c}_1\,\mathrm{e}^{-\tilde{\gamma}T_k})\\ &\leq \mathrm{e}^{k'_u\tau}(c_0 + c_1\,\mathrm{e}^{-\gamma t}) + k'_c\,\mathrm{e}^{k'_u\tau}\tau(\tilde{c}_0 + \tilde{c}_1\,\mathrm{e}^{-\tilde{\gamma}t})\end{aligned}$$

With this inequality, we go back to the derivative of $V_i(t)$ and resort Lemma 9.8 in Khalil (2002). Thus, $V_i(t)$ exponentially converges to a neighbourhood of the origin with its radius proportional to $c_0 + \tilde{c}_0$. This together with the convergence of $W_0(t)$ completes the proof. ■

Remark 3.3: Theorems 3.1 and 3.2 propose two different event-triggered optimal consensus algorithms for multi-agent systems (1). Compared with existing continuous or event-triggered optimal consensus results, we consider more general agent dynamics. Moreover, we do not require the controllers to instantly react to the received neighbouring information as that in Kia et al. (2015), S. Liu et al. (2016), Deng et al. (2016), Yi et al. (2018), Z. Li et al. (2020), and Yu and Chen (2021). Both the information sharing communication and controller update happen only if necessary according to the proposed event-triggered mechanisms. Thus, the proposed optimal consensus designs will definitely save many communication and computation resources.

4. Simulation

In this section, we provide two examples to illustrate the effectiveness of our event-triggered optimal consensus controllers.

Example 4.1: Consider a group of four mobile robotics modelled by double integrator dynamics $\ddot{y}_i = u_i$ with position $y_i \in \mathbb{R}^2$ and input $u_i \in \mathbb{R}^2$. We want to drive all robots to rendezvous at a common point that minimises the aggregate distance from their starting points to this final location. The communication graph among these robots is depicted as Figure 1 with all the edge weights as 1.

For this purpose, we assign $f_i(s) = \|s - y_i(0)\|^2$ to robot i and let $f(s) \triangleq \sum_{i=1}^4 \|s - y_i(0)\|^2$. Then the problem is converted to an optimal output consensus problem for linear multi-agent systems (1) with system matrices as follows:

$$A_i = \begin{bmatrix}\mathbf{0} & \mathbb{I}_2\\ \mathbf{0} & \mathbf{0}\end{bmatrix},\quad B_i = \begin{bmatrix}\mathbf{0}\\ \mathbb{I}_2\end{bmatrix},\quad C_i = \begin{bmatrix}\mathbb{I}_2 & \mathbf{0}\end{bmatrix}$$

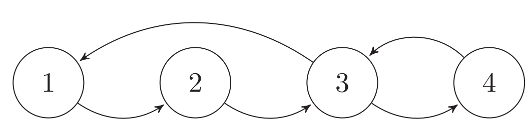

Figure 1. Communication digraph $\mathcal{G}$ in our examples.

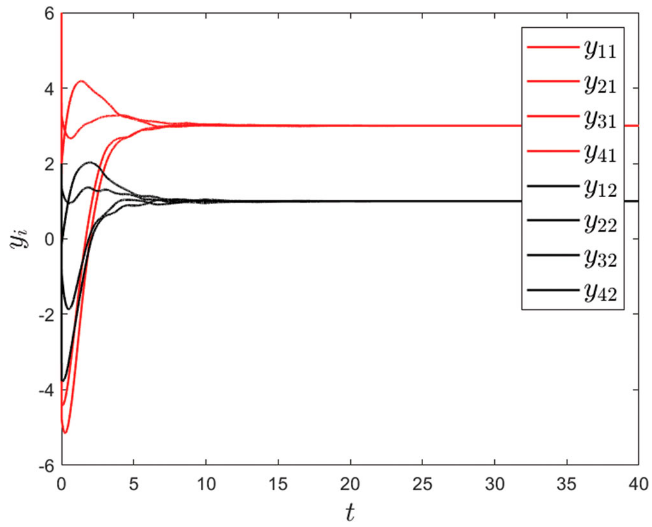

Figure 2. Profile of y_i under the controller (3) in **Example 4.1**.

Since Assumptions 2.1–3.1 can be easily confirmed, we resort to Theorem 3.1 and develop distributed algorithms of the form (3) with rules (5) and (6) to solve the problem.

In simulations, we choose $K_{i1} = [-\mathbb{I}_2 \;\; -2\mathbb{I}_2], K_{i2} = \mathbb{I}_2, \alpha = 1$, $\beta = 10$, $c_0 = \tilde{c}_0 = 0$, $c_1 = \tilde{c}_1 = 3$, $\gamma = 0.5$, and $\tilde{\gamma} = 0.3$ for the event-triggered algorithm. Set $y(0) = [0\ 0\ 6\ 0\ 6\ 2\ 0\ 2]^\top$ and all other initial conditions are randomly generated. We list the profiles of y_i in Figure 2. It can be found that all robots indeed achieve a rendezvous at the optimal position $y^* = [3\ 1]^\top$. Moreover, we count the numbers of information sharing and control update at each agent between 0 s and 40 s. They are (208, 194, 330, 141) and (30, 46, 40, 44). Apparently, the numbers of control update are considerably reduced compared with the case when we mainly focus on event-triggered communication. This verifies the efficiency of algorithm (3) to save such computation resources.

Example 4.2: We further consider another example to show that the presented event-triggered designs can be applied to more general multi-agent systems with complex cost functions. Suppose that the system matrices are heterogeneous as follows:

$$A_1 = 1, \quad B_1 = 1, \quad C_1 = 1$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^\top$$

$$A_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^\top$$

$$A_4 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^\top$$

They have the same communication graph in **Example 1**. The cost functions are chosen as $f_1(s) = \frac{1}{2}(s-2)^2, f_2(s) = s^2 \ln(1 + s^2) + (s+1)^2$, $f_3(s) = \ln(e^{-0.1s} + e^{0.3s}) + s^2$, and $f_4(s) = \frac{s^2}{25\sqrt{s^2+1}} + (s-3)^2$. By some mathematical manipulations, Assumptions 2.1–3.1 can be practically verified. Here we resort to Theorem 3.2 to solve the problem by periodic checking event-triggered algorithm (15).

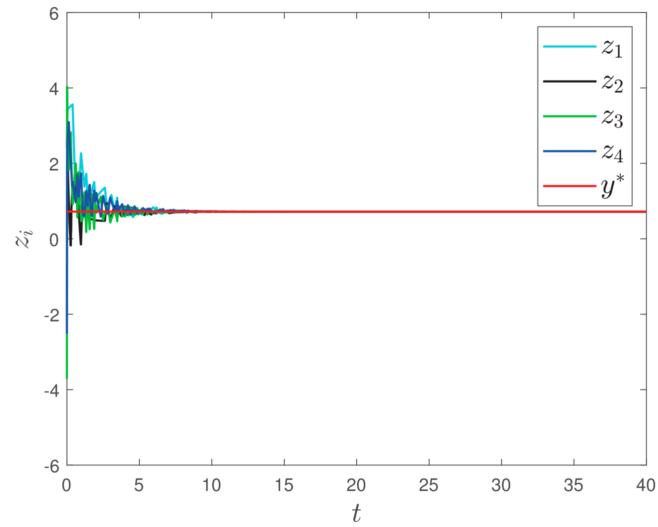

Figure 3. Profile of z_i under the controller (15) in **Example 4.2**.

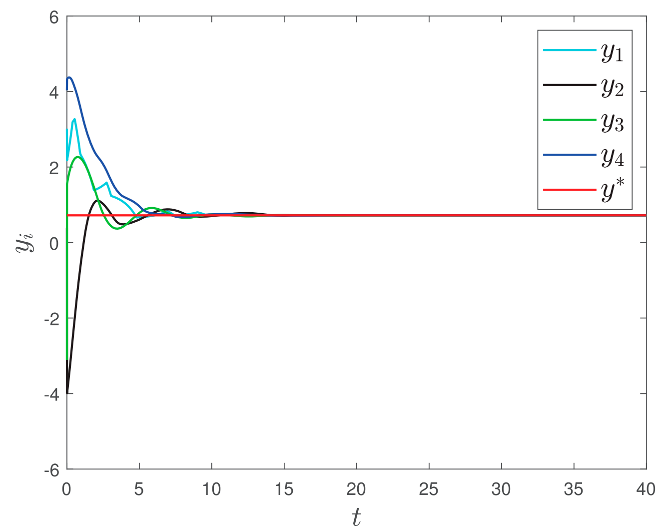

Figure 4. Profile of y_i under the controller (15) in **Example 4.2**.

In the simulation, we set the control gain matrices as $K_{11} = -2.4142$, $K_{12} = 1.4142$, $K_{21} = [-0.4142 \;\; -1.3522]$, $K_{22} = 1.4142$, $K_{31} = [-2.7331 \;\; -2.3372 \;\; -3.5835]$, $K_{32} = 3.3166$, $K_{41} = [-1.0000 - 1.7321]$, and $K_{42} = 1.0000$. We use the same α, β, c_0, $\tilde{c}_0$, c_1, $\tilde{c}_1$, γ, and $\tilde{\gamma}$ for (15), and set $\tau = 0.02$. To implement the algorithm, we use the forward Euler method with step size $\tau_e = 0.002$ s. We list the profiles of z_i, y_i, and u_i in Figures 3–5. It can be found that the outputs of all agents reach the expected consensus on the global optimal solution quickly while the control efforts are bounded and converge to some steady-states. We also count the numbers of information sharing and control update at each agent between 0 s and 40 s. Compared with the traditional periodical treatment with $t_k^i = \tilde{t}_k^i = k\tau_e$ where 4000 times of information sharing communication and 2000 times of controller update are required, such numbers by our event-triggered rule (15) for each agent are only (111, 87, 242, 75) and (27, 30, 76, 26), which have been substantially reduced. These observations confirm the efficiency of our event-based designs to solve the optimal output consensus problem for these heterogeneous high-order multi-agent systems.

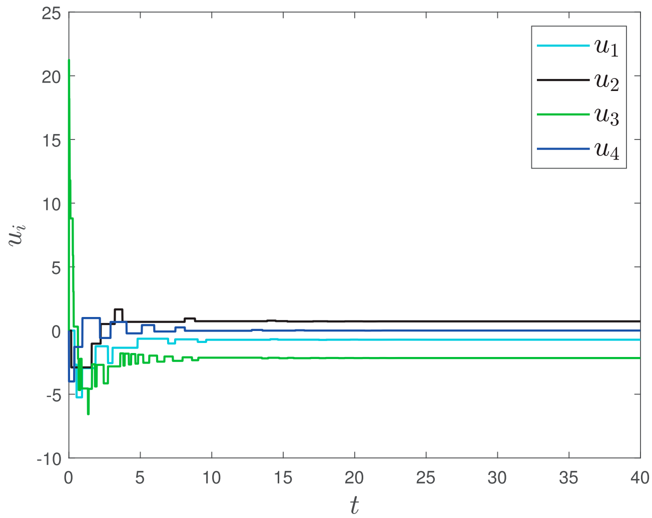

Figure 5. Profile of u_i under the controller (15) in **Example 4.2**.

5. Conclusion

We have considered the optimal consensus problem for a group of heterogeneous linear multi-agent systems. Two Zeno-free distributed algorithms were provided relying on event-triggered controller update and event-triggered communication. The agents' outputs are shown to reach an approximate or exact optimal consensus on the expected optimal point specified by the given optimisation problem under different parameters. In the future, we may consider how to extend these designs to the fully distributed case for nonlinear multi-agent systems or consider output feedback designs.

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