Distributed Nash Equilibrium Seeking Dynamics With Discrete Communication

Rui Yu, Yutao Tang, Peng Yi, and Li Li

***Abstract*— In this brief, we aim to provide a distributed Nash equilibrium seeking algorithm in continuous time with discrete communications. A group of agents are considered playing a continuous-kernel noncooperative game over a network. The agents need to seek the Nash equilibrium when each player cannot get the overall action profiles in real time, rather are only able to get information from its networked neighbors. Meanwhile, a continuous-time dynamics is discussed for the players to update their variables, but the communications over the network are only assumed to allow at discrete-time instants, since continuous-time communications are prohibitive and cumbersome in practice. First, the periodic communication is considered at a fixed interval, and the solvability of Nash equilibrium seeking is shown with discrete communications. Then, an event-trigger communication scheme is proposed to further reduce the communication rounds. Nevertheless, the event-trigger communication scheme requires each player continuously monitoring its local states. To alleviate the monitoring burden, a periodic event detection mechanism is further developed. The exponential convergence of the dynamics with the three discrete communication schemes is proven. Finally, the comparative simulation studies are designed to illustrate the algorithm performance with different communication schemes and parameter settings.**

***Index Terms*— Discrete-time communication, event trigger, Nash equilibrium seeking, periodic communication.**

I. Introduction

Distributed Nash equilibrium seeking for noncooperative games has been a hot topic during the past few years due to its wide applications [1], [2], [3], [4]. Each player has a private cost function depending its own and other players' actions and pursues at minimizing its own cost function. Plenty of results have been delivered in this topic for different game types [5], [6], [7], [8], [9], [10], [11], [12], [13], [14].

Recently, continuous-time Nash equilibrium seeking algorithms have drawn significant attention due to the fact that many Nash equilibrium seeking problems can depend on or be implemented by continuous-time physical plants. For example, a consensus-based design idea was utilized in [11] to develop continuous-time gradient-play algorithms to see the Nash equilibrium of some noncooperative games. By fully exploiting the passivity property of the presented algorithm dynamics, the authors showed the effectiveness of their design under connected undirected communication graphs. A similar consensus-based protocol was also proposed in [15] along with both local and nonlocal stability analyses of the algorithms under different function conditions. Distributed averaging integral algorithms were further developed in [16] to remove the strong coupling condition required in [11] via the dynamic gain mechanism. Meanwhile, some interesting continuous-time designs have been delivered to extend these algorithms to the case with directed or time-varying graphs [17], [18], [19], [20], [21].

Note that all the above continuous-time Nash equilibrium seeking algorithms require the real-time decision information from its neighbors. This implies that the agents must continuously observe or share information with others through the underlying communication network, which costs too much and can be prohibitive in implementation. At the same time, various multiagent designs with both time-triggered and event-triggered communications have been derived in the literature (see a survey paper [22]). Inspired by these achievements [23], [24], [25], [26], [27], it is natural to ask whether continuous-time communication is necessary for an effective continuous-time Nash equilibrium seeking algorithm. If not, we are interested in distributed continuous-time Nash equilibrium seeking algorithms with different discrete-time communications to save communication resources. During the revision process, we have noticed some recent works [28], [29] on this issue for some specific settings. However, there has been no comprehensive discussion of introducing different discrete-time communication mechanisms in distributed continuous-time Nash equilibrium seeking algorithms.

With these questions in mind, we consider a noncooperative game played by a group of continuous-time agents with discrete communications. That is, each player can continuously update the action to minimize its own cost function, while the information sharing is only allowed at specified discrete-time instants. Note that with this type of discrete-time communications, the gathered neighboring information by agents is naturally subject to multiple time delays and jumps. Thus, the convergent performance of the existing continuous-time Nash equilibrium seeking algorithms is probably deteriorated and fails to work in these circumstances. Furthermore, the associated closed-loop system is basically hybrid, which makes both analysis and design of our problem more challenging than its pure continuous-time or discrete-time counterpart.

Motivated by the desire of combing existing continuous-time results with discrete-time communications, we adopt the distributed algorithm given in [17] as our starting point and discuss its implementation with different kinds of discrete-time communications to solve the formulated problem understanding assumptions.

The contributions of this brief are summarized as follows.

1) We formulate and solve the continuous-time Nash equilibrium seeking problem under discrete communications. Compared with the existing continuous-time results, the developed algorithms remove the continuous communication requirement among the agents, which can save the communication resources and enlarge the potential applications of the existing Nash equilibrium seeking designs.

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2) We systematically develop three different effective schemes to determine the discrete instants of communication times upon different needs. For each case, we constructively present some sufficient parameter choices. All schemes are rigorously proven to be Zeno-free and retain an exponential convergence rate to solve the formulated problem.

The rest of this brief is organized as follows. We first give some preliminaries in Section II and formulate the problem in Section III. Then, main results with different kinds of communication schemes are provided in Section IV. Following that, the comparative simulation studies are presented in Section V. Finally, the conclusions are given in Section VI.

II. Preliminaries

Denote by $\mathbb{R}^n$ the n-dimensional Euclidean space. Let $\text{diag}\{b_1, \ldots, b_n\}$ be the n-dimensional diagonal matrix with $b_1, \ldots, b_n$ on the main diagonal and $\text{col}(a_1, \ldots, a_n) \triangleq [a_1^\top, \ldots, a_n^\top]^\top$. $\otimes$ represents the Kronecker product. Denote by $\|x\|$ the Euclidean norm of a vector $x \in \mathbb{R}^n$ and by $\|A\|$ the spectral norm of a matrix $A \in \mathbb{R}^{n\times n}$. Let $\mathbf{1}_n$ (or $\mathbf{1}_{n\times m}$) and $\mathbf{0}_n$ (or $\mathbf{0}_{n\times m}$) represent the n-dimensional vector (or $n \times m$ matrix) with all elements being 1 and 0, respectively.

A function $f : \mathbb{R}^m \to \mathbb{R}$ is said to be convex if $f(\varepsilon x_1+(1-\varepsilon)x_2) \le \varepsilon f(x_1) + (1-\varepsilon) f(x_2)$ for any $0 \le \varepsilon \le 1$ and any $x_1, x_2 \in \mathbb{R}^m$. We say f is strictly convex if the above inequality is strict for any $0 < \varepsilon < 1$ when $x_1 \ne x_2$. For a vector-valued function $\Phi: \mathbb{R}^m \to \mathbb{R}^m$, we say it is ω-strongly monotone with some constant $\omega > 0$ if $(x_1 - x_2)^\top[\Phi(x_1) - \Phi(x_2)] \ge \omega\|x_1 - x_2\|^2$ for any $x_1, x_2 \in \mathbb{R}^m$. This vector-valued function Φ is said to be ϑ-Lipschitz with some constant $\vartheta > 0$ when $\|\Phi(x_1) - \Phi(x_2)\| \le \vartheta\|x_1 - x_2\|$ for any $x_1, x_2 \in \mathbb{R}^m$. More details can be found in [30].

We use weighted directed graphs to describe the information sharing structure in a multiagent system [31]. A weighted directed graph (digraph) is represented by a triplet $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$ with the node set $\mathcal{N} = \{1, \ldots, N\}$, the edge set $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$, and the adjacency matrix $\mathcal{A} = [a_{ij}]_{N\times N}$. A directed path in this graph is an alternating sequence of nodes and edges. If there is a directed path between any two nodes in $\mathcal{G}$, we say this digraph is strongly connected. The neighbor set of player i is defined as $\mathcal{N}_i = \{j \mid (j, i) \in \mathcal{E}\}$. $d_i^{\text{in}} = \sum_{j=1}^N a_{ij}$ represents the in-degree of node i, $d_i^{\text{out}} = \sum_{j=1}^N a_{ji}$ represents the out-degree of node i, and $\mathcal{D} = \text{diag}(d_1^{\text{in}}, \ldots, d_N^{\text{in}})$. A directed graph is weight balanced if $d_i^{\text{in}} = d_i^{\text{out}}$ holds for any $i \in \mathcal{N}$. The Laplacian matrix of $\mathcal{G}$ is defined as $L \triangleq \mathcal{D} - \mathcal{A}$. When a directed graph is weight balanced, we have $L\mathbf{1}_N = \mathbf{0}_N$ and $\mathbf{1}_N^T L = \mathbf{0}_N^T$. Consider a matrix $\text{Sym}(L) \triangleq ((L + L^T)/2)$. It is positive semidefinite with ordered eigenvalues $0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$. Moreover, $\lambda_2 > 0$, if and only if this digraph is strongly connected. In this case, let $M_1 = (1/\sqrt{N})\mathbf{1}_N$ and $M_2 \in \mathbb{R}^{N\times(N-1)}$ to satisfy $M_2^T M_1 = \mathbf{0}_{N-1}$, $M_2^T M_2 = I_{N-1}$, and $M_2 M_2^T = I_N - M_1 M_1^T$. We have $\lambda_2 I_{N-1} \le M_2^T \text{Sym}(L) M_2 \le \lambda_N I_N$.

III. Problem Formulation

Consider a noncooperative game played by N agents described as follows. Each agent $i \in \mathcal{N} \triangleq \{1, \ldots, N\}$ has an action $z_i \in \mathbb{R}$ and a local cost function $J_i(z_i, z_{-i})$ depending upon both its own and the others' actions with $z_{-i} \triangleq (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_N) \in \mathbb{R}^{N-1}$. In this game, each player selfishly pursues in minimization of its own cost function by continuously updating its own action profile z_i.

Denote this noncooperative game by $\text{G} \triangleq \{\mathcal{N}, J_i, \mathbb{R}\}$. Its Nash equilibrium is defined as follows [32].

Definition 1: An action vector $z^\star = \text{col}(z_1^\star, \ldots, z_N^\star) \in \mathbb{R}^N$ is called a Nash equilibrium of game $\text{G} \triangleq \{\mathcal{N}, J_i, \mathbb{R}\}$ if $J_i(z_i^\star, z_{-i}^\star) \le J_i(z_i, z_{-i}^\star)$ for any $i \in \mathcal{N}$ and $z_i \in \mathbb{R}$.

It is observed that the cost function of each agent will not be improved by a unilateral change in its own action at a Nash equilibrium. For simplicity, we denote $F(y) \triangleq \text{col}(\nabla_1 J_1(z_1, z_{-1}), \ldots, \nabla_N J_N(z_N, z_{-N})) \in \mathbb{R}^N$ with $\nabla_i J_i(z_i, z_{-i}) \triangleq (\partial/\partial z_i) J_i(z_i, z_{-i}) \in \mathbb{R}$. Here, F is called the pseudogradient associated with $J_1, \ldots, J_N$.

To guarantee the well posedness of our problem, we make the following assumptions as in [11], [16], and [17].

Assumption 1: For any $i \in \mathcal{N}$, the function $J_i(z_i, z_{-i})$ is twice continuously differentiable, strictly convex, and radially unbounded in $z_i \in \mathbb{R}$ for any fixed $z_{-i} \in \mathbb{R}^{N-1}$.

Assumption 2: The pseudogradient F is $\underline{l}$-strongly monotone and $\bar{l}$-Lipschitz for two constants $\underline{l}, \bar{l} > 0$, that is, for any $x_1, x_2 \in \mathbb{R}^N$

$$(x_1 - x_2)^\top[F(x_1) - F(x_2)] \ge \underline{l}\|x_1 - x_2\|^2$$
$$\|F(x_1) - F(x_2)\| \le \bar{l}\|x_1 - x_2\|.$$

Under Assumptions 1 and 2, we can resort to [33, Propositions 1.4.2 and 2.2.7] and conclude that game G admits a unique Nash equilibrium $z^\star = \text{col}(z_1^\star, \ldots, z_N^\star)$, which satisfies the equation $F(z^\star) = \mathbf{0}$.

Note that the cost function depends upon both z_i and z_{-i}. To reach the Nash equilibrium, players are required to get the others' actions for update rules. Here, we are interested in the partial-information case, where each player cannot get the overall action profiles in real time. Meanwhile, they can communicate with its immediate neighbors through a communication network described by a graph $\mathcal{G}$ and estimate other players' actions. Denote by z_j^i agent i's estimate of agent j's strategy. Define $\mathbf{z}^i = \text{col}(z_1^i, \ldots, z_N^i) \in \mathbb{R}^N$ with $z_i^i = z_i$ and $\nabla_i J_i(\mathbf{z}^i) = (\partial J_i/\partial z_i^i)(\mathbf{z}^i)$ as the partial gradient of agent i's cost function evaluated at its estimate $\mathbf{z}^i$. For convenience, an extended pseudogradient as $\boldsymbol{F}(\mathbf{z}) = \text{col}(\partial_1 J_1(\mathbf{z}^1), \ldots, \partial_N J_N(\mathbf{z}^N)) \in \mathbb{R}^N$ is defined for this game.

This Nash equilibrium seeking problem has been partially studied in past few years, and many interesting continuous-time algorithms have been derived. In particular, continuous-time algorithms incorporating consensus-based estimators have been paid much attention to in [11], [16], and [17]. Such designs boil down to construct a continuous-time algorithm of the following form:

$$\begin{aligned} \dot{z}_i &= g_i^C\big(z_i, z_k^i, z_i^j\big) \\ \dot{z}_k^i &= h_{ik}^C\big(z_i, z_k^i, z_k^j\big), \quad j \in \mathcal{N}_i,\ k \in \mathcal{N}\backslash\{i\} \end{aligned} \tag{1}$$

where g_i^C and h_{ik}^C are chosen smooth functions, such that z_i can converge to the expected Nash equilibrium $z_i^\star$.

However, distributed algorithms (1) require a continuous information flow in the multiagent system. That is, each player has to continuously communicate with its neighbors and share the real-time information. This requirement is definitely of a high cost in communication resources for networked systems. To save resources, we aim at continuous-time Nash equilibrium seeking algorithms with discrete-time communications for players to reach the expected Nash equilibrium.

For this purpose, we suppose each agent is only allowed to push its own information to its neighbors through the underlying communication network at some discrete-time instants. Let $0 = t_0^i < t_1^i < \cdots$ be the time instants. The pushed action profile by player i during the time interval $[0, +\infty)$ is denoted by $\hat{\mathbf{z}}^i(t) = \text{col}(\hat{z}_1^i(t), \ldots, \hat{z}_N^i(t))$, which is given by $\hat{z}_j^i(t) = z_j^i(t_k^i)$ for each $t \in [t_k^i, t_{k+1}^i)$. Note that each player i can continuously update its own action $z_i(t)$ and recompute $z_i(t)$ immediately whenever player i receives a new action profile from one of its neighbors.

Then, our Nash equilibrium seeking problem with discrete communications is formulated as follows: Given function J_i and graph $\mathcal{G}$, find smooth functions g_i^C and h_{ik}^C and communication time instants

$\{t_k^i\}$, such that, from any initial condition, the trajectory of the following dynamic system:

$$\begin{aligned}\dot{z}_i &= g_i^C\big(z_i,\, z_k^i,\, \hat{z}_i^j\big)\\ \dot{z}_k^i &= h_{ik}^C\big(z_i,\, z_k^i,\, \hat{z}_k^j\big), \quad j\in\mathcal{N}_i,\ k\in\mathcal{N}\backslash\{i\}\end{aligned} \tag{2}$$

is well defined over the time interval $[0, +\infty)$ and satisfies that $\lim_{t\to+\infty} z_i(t) = z_i^\star$ with $z^\star \triangleq \mathrm{col}(z_1^\star,\ldots,z_N^\star)$ being a Nash equilibrium of game $\mathrm{G} \triangleq \{\mathcal{N},\, J_i,\, \mathbb{R}\}$.

Compared with (1), the received information from its neighbors by each agent in (2) is not real time and subject to some time delays determined by the time instants $\{t_k^i\}$. The main goal of this brief is to develop effective distributed Nash equilibrium seeking algorithms in continuous time for different discrete communication settings.

Before the main results, we list two assumptions to ensure the solvability of our problem.

Assumption 3: Digraph $\mathcal{G}$ is weight balanced and strongly connected.

Assumption 4: The extended pseudogradient $\boldsymbol{F}$ is l_F-Lipschitz with $l_F > 0$, that is, for any $x_1,\, x_2 \in \mathbb{R}^{N^2}$

$$\|\boldsymbol{F}(x_1) - \boldsymbol{F}(x_2)\| \le l_F\|x_1 - x_2\|.$$

Assumption 3 provides a sufficient connectivity of the communication network, and Assumption 4 on the extended pseudogradient $\boldsymbol{F}$ extends the Lipschitz continuity of F to the corresponding augmented space. These two assumptions have been widely made in the literature [11], [16], [17].

IV. Main Result

In this section, we will first present a periodic mechanism to demonstrate the validity of distributed continuous-time algorithm with discrete-time communications and then provide more efficient event-triggered schemes to save both communication and computation resources.

Motivated by the gradient-play rules in [11], [15], and [17], we are interested in the following dynamics to solve our problem:

$$\begin{aligned}\dot{z}_i &= -\alpha\sum_{j=1}^N a_{ij}\big(z_i - \hat{z}_i^j\big) - \nabla_i J_i(\mathbf{z}^i)\\ \dot{z}_k^i &= -\alpha\sum_{j=1}^N a_{ij}\big(z_k^i - \hat{z}_k^j\big), \quad k\in\mathcal{N}\backslash\{i\}\end{aligned} \tag{3}$$

where $\hat{z}_i^j$ is the pushed version of z_i^j by agent j to its neighbors, and the constant $\alpha > 0$ is to be specified later.

Letting $R_i = \mathrm{col}(\mathbf{0}_{i-1},\, 1,\, \mathbf{0}_{N-i})$ gives

$$\dot{\mathbf{z}}^i = -\alpha\sum_{j=1}^N a_{ij}(\mathbf{z}^i - \hat{\mathbf{z}}^j) - R_i\nabla_i J_i(\mathbf{z}^i), \quad j\in\mathcal{N}\backslash\{i\}$$

with $\hat{\mathbf{z}}^j$ is the pushed version of $\mathbf{z}^j$ by agent j.

Denoting $\boldsymbol{e}_i(t) = \hat{\mathbf{z}}^i(t) - \mathbf{z}^i(t)$ and $\boldsymbol{e} = \mathrm{col}(\boldsymbol{e}_1,\ldots,\boldsymbol{e}_N)$, we can further put it into a compact form

$$\dot{\mathbf{z}} = -\alpha\boldsymbol{L}\hat{\mathbf{z}} + \alpha\boldsymbol{A}\boldsymbol{e} - R\boldsymbol{F}(\mathbf{z})$$

with $R = \mathrm{diag}(R_1,\ldots,R_N)$, $\mathbf{z} = \mathrm{col}(\mathbf{z}^1,\ldots,\mathbf{z}^N)$, $\hat{\mathbf{z}} = \mathrm{col}(\hat{\mathbf{z}}^1,\ldots,\hat{\mathbf{z}}^N)$, $\boldsymbol{L} = L\otimes I_N$, and $\boldsymbol{A} = \mathcal{A}\otimes I_N$.

A. Solvability With Periodic Communication

In this section, we start with the periodic communication case to confirm the effectiveness of algorithm (3).

We suppose all players will push their information to the neighbors through the communication network periodically at $t_k^i = k\tau$ for a constant $\tau > 0$ to be specified later.

Let $c_{\max} = \max\{\|\mathcal{A}\|, \|L\|\}$ and $l = \max\{\bar{l},\, l_F\}$, and consider the symmetric matrix

$$A_\alpha = \begin{bmatrix} \dfrac{\underline{l}}{N} & -\dfrac{l}{\sqrt{N}}\\ -\dfrac{l}{\sqrt{N}} & \alpha\lambda_2 - l\end{bmatrix}.$$

It is verified to be positive definite when $\alpha > (1/\lambda_2)((l^2/\underline{l}) + l)$. In this case, we denote by ν the minimal eigenvalue of A_α.

Theorem 1: Suppose Assumptions 1–3 hold and $\alpha > (1/\lambda_2)((l^2/\underline{l}) + l)$. Then, the Nash equilibrium seeking problem is exponentially solved by algorithm (3) with $t_k^i = k\tau$ if

$$0 < \tau \le \frac{\nu}{(2\alpha c_{\max} + l)(2\alpha c_{\max} + \nu)}. \tag{4}$$

Proof: Denoting $\mathbf{z}^\star = \mathbf{1}\otimes z^\star$ and $\tilde{\mathbf{z}} = \mathbf{z} - \mathbf{z}^\star$, we perform the coordinate transformation

$$\bar{\mathbf{z}}_1 = \big(M_1^T\otimes I_N\big)\tilde{\mathbf{z}}, \quad \bar{\mathbf{z}}_2 = \big(M_2^T\otimes I_N\big)\tilde{\mathbf{z}}.$$

It follows then:

$$\begin{aligned}\dot{\bar{\mathbf{z}}}_1 &= \alpha\big[\big(M_1^T\mathcal{A}\big)\otimes I_N\big]\boldsymbol{e} - \big(M_1^T\otimes I_N\big)R\,\triangle\\ \dot{\bar{\mathbf{z}}}_2 &= -\alpha\big[\big(M_2^T L M_2\big)\otimes I_N\big]\bar{\mathbf{z}}_2 - \big(M_2^T\otimes I_N\big)R\,\triangle\\ &\quad + \alpha\big[\big(M_2^T\mathcal{A}\big)\otimes I_N\big]\boldsymbol{e} - \alpha\big[\big(M_2^T L\big)\otimes I_N\big]\boldsymbol{e}\end{aligned} \tag{5}$$

where $\triangle \triangleq \boldsymbol{F}(\mathbf{z}) - \boldsymbol{F}(\mathbf{z}^\star)$.

To prove this theorem, we use t_k short for t_k^i without confusions and select a quadratic Lyapunov function candidate for system (5) as follows:

$$V(\bar{\mathbf{z}}_1, \bar{\mathbf{z}}_2) = \frac{1}{2}(\|\bar{\mathbf{z}}_1\|^2 + \|\bar{\mathbf{z}}_2\|^2).$$

During the interval $[t_k,\, t_{k+1})$, its time derivative along the trajectory of system (5) is given as follows:

$$\begin{aligned}\dot{V} &= -\tilde{\mathbf{z}}^T R\,\triangle - \alpha\bar{\mathbf{z}}_2^T[(M_2^T\mathrm{Sym}(L)M_2)\otimes I_N]\bar{\mathbf{z}}_2\\ &\quad + \alpha\tilde{\mathbf{z}}^T\boldsymbol{A}\boldsymbol{e} - \alpha\bar{\mathbf{z}}_2^T[(M_2^T L)\otimes I_N]\boldsymbol{e}\\ &\le -\tilde{\mathbf{z}}^T R\,\triangle - \alpha\lambda_2\|\bar{\mathbf{z}}_2\|^2 + \alpha c_{\max}\|\tilde{\mathbf{z}}\|\|\boldsymbol{e}\| + \alpha c_{\max}\|\bar{\mathbf{z}}_2\|\|\boldsymbol{e}\|\\ &\le -\big[\|\bar{\mathbf{z}}_1\| \quad \|\bar{\mathbf{z}}_2\|\big]A_\alpha\begin{bmatrix}\|\bar{\mathbf{z}}_1\|\\ \|\bar{\mathbf{z}}_2\|\end{bmatrix} + \alpha c_{\max}\|\tilde{\mathbf{z}}\|\|\boldsymbol{e}\|\\ &\quad + \alpha c_{\max}\|\bar{\mathbf{z}}_2\|\|\boldsymbol{e}\|\\ &\le -2\nu V + \alpha c_{\max}\|\tilde{\mathbf{z}}\|\|\boldsymbol{e}\| + \alpha c_{\max}\|\bar{\mathbf{z}}_2\|\|\boldsymbol{e}\|\end{aligned}$$

where we use the fact $\|M_2\| = \|R\| = 1$ by definition and handle the term $-\tilde{\mathbf{z}}^T R\triangle$ as that in [17].

Then, by Young's inequality, we have

$$\dot{V} \le -\frac{1}{2}\nu V + \left(\frac{2\alpha^2 c_{\max}^2}{\nu}\|\boldsymbol{e}\|^2 - \frac{1}{2}\nu V\right). \tag{6}$$

We are going to show that any chosen τ satisfying condition (4) can ensure $((2\alpha^2 c_{\max}^2)/\nu)\|\boldsymbol{e}\|^2 \le (1/2)\nu V$, or equivalently, $(\|\boldsymbol{e}\|/\|H\|) \le (\nu/(2\alpha c_{\max}))$ with $H = \mathrm{col}(\bar{\mathbf{z}}_1, \bar{\mathbf{z}}_2)$ during each time interval $[t_k, t_{k+1})$.

To this end, we denote $y(t) = (\|\boldsymbol{e}\|/\|H\|)$ and take its derivative during this interval and obtain that

$$\begin{aligned}\frac{d}{dt}\frac{\|\boldsymbol{e}\|}{\|H\|} &= \frac{d}{dt}\frac{(\boldsymbol{e}^T\boldsymbol{e})^{\frac{1}{2}}}{(H^T H)^{\frac{1}{2}}}\\ &= \frac{(\boldsymbol{e}^T\boldsymbol{e})^{-\frac{1}{2}}\boldsymbol{e}^T\dot{\boldsymbol{e}}(H^T H)^{\frac{1}{2}} - (H^T H)^{-\frac{1}{2}}H^T\dot{H}(\boldsymbol{e}^T\boldsymbol{e})^{\frac{1}{2}}}{H^T H}\\ &= \frac{\boldsymbol{e}^T\dot{\boldsymbol{e}}}{\|\boldsymbol{e}\|\|H\|} - \frac{\|\boldsymbol{e}\|H^T\dot{H}}{\|H\|^3}\end{aligned} \tag{7}$$

where we use the facts that $\dot{\boldsymbol{e}}(t) = -\dot{\mathbf{z}}(t) = -\dot{\tilde{\mathbf{z}}}(t)$ and $\tilde{\mathbf{z}}(t) = (M_1 \otimes I_N)\bar{\mathbf{z}}_1(t) + (M_2 \otimes I_N)\bar{\mathbf{z}}_2(t)$. Note that

$$\begin{aligned}\|\dot{\boldsymbol{e}}(t)\| &= \|\dot{\tilde{\mathbf{z}}}(t)\| \le \|\dot{H}(t)\| \\ &\le l\|\bar{\mathbf{z}}\| + \alpha c_{\max}\|\bar{\mathbf{z}}_2\| + 2\alpha c_{\max}\|\boldsymbol{e}(t)\| \\ &\le (2\alpha c_{\max} + l)(\|H(t)\| + \|\boldsymbol{e}(t)\|).\end{aligned}$$

This together with (7) implies that

$$\dot{y} \le (1+y)\frac{\|\dot{H}\|}{\|H\|} \le (2\alpha c_{\max} + l)(1+y)^2.$$

Let us consider the differential equation $\dot{\phi} = (2\alpha c_{\max} + l)(1+\phi)^2$ with $\phi(t_0) = 0$. Its solution is $\phi(t, t_0) = ((2\alpha c_{\max} + l)(t - t_0))/(1 - (2\alpha c_{\max} + l)(t - t_0))$. Since $y(t_k, t_k) = 0$, we recall the comparison lemma in [34] and conclude that

$$y(t, t_k) \le \frac{(2\alpha c_{\max} + l)(t - t_k)}{1 - (2\alpha c_{\max} + l)(t - t_k)}.$$

Under condition (4), it follows for $t \in [t_k, t_{k+1})$ that:

$$\frac{\|\boldsymbol{e}(t)\|}{\|H(t)\|} \le y(t_k + \tau, t_k) \le \frac{(2\alpha c_{\max} + l)\tau}{1 - (2\alpha c_{\max} + l)\tau} = \frac{\nu}{2\alpha c_{\max}}.$$

This jointly with (6) provides us

$$\dot{V} \le -\frac{\nu}{2}V$$

for any $t \in [0, \infty)$. Thus, according to [34, Th. 4.10], we can conclude that $\bar{\mathbf{z}}(t) \to 0$ as $t \to \infty$, or equivalently, $z^i(t)$ exponentially converges to $z^\star$ as $t \to \infty$. The proof is complete.

Remark 1: This theorem gives an affirmative answer that the Nash equilibrium seeking problem is indeed solvable by continuous-time algorithms with discrete-time communications. Compared with plenty of continuous-time algorithms in the literature, this rule (3) does not require agents to share their information continuously, which definitely reduces traffic on the communication network.

Remark 2: The criterion for selecting control gain α and periodic constant τ reveals the natural trade-off between the control effort and graph algebraic connectivity. Note that the choice of α and τ in this theorem requires some global knowledge, e.g., λ_2 and $c_{\max}$. In practice, we may estimate them beforehand to implement this algorithm [35].

B. Solvability With Event-Triggered Communication

In this section, we consider event-triggered aperiodic communication schemes to solve our problem. In contrast to the previous periodic scheme, event-triggered schemes may result in a more efficient use of the resources.

Motivated by the developed designs in [22] and [36], we introduce a triggering function as follows:

$$f_i(t,\ \boldsymbol{e}_i(t)) = \|\boldsymbol{e}_i(t)\| - (c_0 + c_1 e^{-\beta t})$$

for player i with parameters c_0, $c_1 \ge 0$, $c_0 + c_1 > 0$, and $\beta > 0$ to be specified later.

During the game, each player i continuously monitors its current state and evaluates the function f_i. It will push its current profile to the neighbors when $f_i(t, \boldsymbol{e}_i(t)) \le 0$ is not satisfied. That is, for a given time t_k^i, the next triggering time can be iteratively determined according to

$$t_{k+1}^i = \inf\big\{t \in \big[t_k^i,\ +\infty\big) \mid f_i(t,\ \boldsymbol{e}_i(t)) > 0\big\}. \tag{8}$$

We present another theorem on the solvability of our problem with event-triggered discrete-time communications.

Theorem 2: Suppose Assumptions 1–3 hold and let $\alpha > (1/\lambda_2)((l^2/\underline{l}) + l)$, $0 < \beta < \nu$. Then, algorithm (3) with communication time instants determined by (8) is free from Zeno behaviors. Moreover, there exists a class $\mathcal{K}$ function γ, such that each $\mathbf{z}^i$ will converge into a ball centered at $z^\star$ with radius $\gamma(c_0)$.

Proof: Since $\|\boldsymbol{e}_i(t)\|$ is enforced to satisfy $f_i(t, \boldsymbol{e}_i(t)) = \|\boldsymbol{e}_i(t)\| - (c_0 + c_1 e^{-\beta t}) \le 0$, the inequality (6) can be strengthened as follows:

$$\dot{V} \le -\nu V + \varpi(c_0 + c_1 e^{-\beta t})^2 \tag{9}$$

where $\varpi = ((2N\alpha^2 c_{\max}^2)/\nu)$.

Solving the differential inequality (9), we have

$$V(t) \le V(0)e^{-\nu t} + \varpi \int_0^t e^{-\nu(t-\tau)}(c_0 + c_1 e^{-\beta t})^2 d\tau. \tag{10}$$

From this, we can easily obtain that there must be a class $\mathcal{K}$ function $\tilde{\gamma}(s) = (\varpi s^2/\nu)$, such that $V(t)$ is ultimately bounded with a tolerance $\tilde{\gamma}(c_0)$. Recalling the definition of V, this implies that each $\mathbf{z}^i$ will converge to a ball center at $z^\star$ with a radius $\gamma(c_0)$ with $\gamma(s) = \sqrt{\tilde{\gamma}(s)}$.

Then, we will rule out the Zeno behaviors. It suffices to show that the length of the interevent intervals is larger than some positive constant τ_0.

Assume that agent i triggers at the time $t^\star$. It follows by (7) that $\boldsymbol{e}_i(t^\star) = 0$ and $f_i(t^\star, \boldsymbol{e}_i(t^\star)) \le 0$. Between two events, the time derivative of $\boldsymbol{e}_i(t)$ is given by

$$\dot{\boldsymbol{e}}_i(t) = \alpha L\boldsymbol{e}_i(t) - \alpha\mathcal{A}\boldsymbol{e}_i(t) + \alpha L\tilde{\mathbf{z}}^i(t) + R_i\nabla_i J_i(\tilde{\mathbf{z}}^i).$$

Using the Lipschitzness of J_i and $\|\boldsymbol{e}_i(t)\| \le (c_0 + c_1 e^{-\beta t})$, there must be positive constants c_2 and c_3, such that

$$\|\dot{\boldsymbol{e}}_i(t)\| \le c_2\|\tilde{\mathbf{z}}^i\| + c_3(c_0 + c_1 e^{-\beta t}). \tag{11}$$

Then, we consider two different cases, i.e., $c_0 \ne 0$ and $c_0 = 0$.

Case I: Suppose $c_0 \ne 0$.

From the boundedness of $\tilde{\mathbf{z}}$, there must be a positive constant c_4, such that

$$\|\dot{\boldsymbol{e}}_i(t)\| \le c_2 c_4 + c_3(c_0 + c_1).$$

This jointly with the fact $\boldsymbol{e}_i(t) = \int_{t^\star}^t \dot{\boldsymbol{e}}_i(s)\mathrm{d}s$ gives

$$\|\boldsymbol{e}_i(t)\| \le \int_{t^\star}^t \|\dot{\boldsymbol{e}}_i(s)\|ds \le [c_2 c_4 + c_3(c_0 + c_1)](t - t^\star).$$

Since the next event is not triggered before $f_i(t, \boldsymbol{e}_i(t)) = \|\boldsymbol{e}_i(t)\| - (c_0 + c_1 e^{-\beta t})$ crosses zero, this does not happen before $\|\boldsymbol{e}_i(t)\| \le c_0$. Thus, a lower bound for the interevent interval is given as

$$\tau_{01} = \frac{c_0}{c_2 c_4 + c_3(c_0 + c_1)} > 0.$$

Case II: Suppose $c_0 = 0$.

In this case, we go back to inequality (11). By (10), we can strengthen it as follows:

$$\|\dot{\boldsymbol{e}}_i(t)\| \le c_5 e^{-\nu t} + c_3 c_1 e^{-\beta t} \le c_5 e^{-\nu t^\star} + c_3 c_1 e^{-\beta t^\star}$$

for some positive constant $c_5 > 0$. Then, we have

$$\begin{aligned}\|\boldsymbol{e}_i(t)\| &\le [c_5 e^{-\nu t^\star} + c_3 c_1 e^{-\beta t^\star}](t - t^\star) \\ &\le e^{-\beta t^\star}[c_5 e^{-(\nu-\beta)t^\star} + c_3 c_1](t - t^\star) \\ &\le e^{-\beta t^\star}[c_5 + c_3 c_1](t - t^\star).\end{aligned}$$

Again, the next event is triggered as soon as $\|\boldsymbol{e}_i(t)\| = c_1 e^{-\beta t}$. We consider the equation $(c_5 + c_3 c_1)s = c_1 e^{-\beta s}$. It has a unique positive root τ_{02}. Moreover, for any $s \in (0, \tau_0 02)$, $(c_5 + c_3 c_1)s < c_1 e^{-\beta s}$. Thus, $\|\boldsymbol{e}_i(t)\| \le e^{-\beta t^\star}(c_5 + c_3 c_1)(t - t^\star) < c_1 e^{-\beta t}$. In other words, $\|\boldsymbol{e}_i(t)\| > c_1 e^{-\beta t}$ will not happen before $t^\star + \tau_{02}$. Thus, $\tau_{02} > 0$ is a lower bound for the interevent interval.

Therefore, we have excluded the possible Zeno behaviors in both cases. This proof is complete.

Remark 3: In contrast to the periodic communications, the event-triggered communication mechanism implementations tie the determination of each player's communication times to their own current states, thus saving network resources more effectively.

C. Solvability With Periodic Event-Triggered Communication

In this section, we propose an enhanced version of the preceding event-triggered scheme where players do not have to monitor their triggering functions continuously. To save space, we only consider the case when $c_0 = 0$.

Motivated by periodic designs in Theorem 1, we assume player i only evaluates function f_i at the time instant $t_k = k\tau$ for a constant $\tau > 0$, that is,

$$t_{k+1}^i = \inf\left\{t_{k'} \middle| t_{k'} > t_k^i \,\&\, f_i(t_{k'}, \boldsymbol{e}_i(t_{k'})) > 0\right\} \tag{12}$$

with $k \in \mathbb{Z}^+$. Similar designs have been developed in [22] to solve the multiagent consensus problem. Compared with rule (2) for continuous-time dynamics, each agent only monitors the triggering condition in a designed time sequence to decide whether to update its pushed action. To this end, the periodic event-triggered scheme can alleviate the monitoring burden through reducing local evaluation.

In this case, the algorithm is naturally free from Zeno behavior. The solvability of our problem under this periodic event-triggered communication is given as follows.

Theorem 3: Suppose Assumptions 1–3 hold. We let $\alpha > (1/\lambda_2)((l^2/\underline{l}) + l)$, $c_0 = 0$, $c_1 > 0$, $0 < \beta < \nu$, and τ satisfying (4). Then, the Nash equilibrium seeking problem is exponentially solved by algorithm (3) with the periodic event-triggered communication specified by (12).

Proof: Similar to the proofs of Theorems 1 and 2, we consider the behavior of $V = (1/2)(||\bar{\mathbf{z}}_1||^2 + ||\bar{\mathbf{z}}_2||^2)$ during each time interval $t \in [t_k^i,\ t_{k+1}^i)$.

By the arguments in Theorem 1, the following inequality holds for any $t \in [t_k^i,\ t_{k+1}^i)$:

$$\dot{V} \leq -\frac{1}{2}\nu V + \left(\frac{2\alpha^2 c_{\max}^2}{\nu}\|\mathbf{e}(t)\|^2 - \frac{1}{2}\nu V\right).$$

In the following, we only have to consider the case when

$$\frac{2\alpha^2 c_{\max}^2}{\nu}\|\mathbf{e}(t)\|^2 \geq \frac{\nu}{2}V$$

holds over this time interval. Otherwise, there must be some time $t' \in [t_k^i,\ t_{k+1}^i)$ at which

$$\frac{2\alpha^2 c_{\max}^2}{\nu}\|\mathbf{e}(t')\|^2 < \frac{\nu}{2}V(\bar{\mathbf{z}}_1(t'),\ \bar{\mathbf{z}}_2(t'))$$

happens. This provides us $\dot{V}(t') \leq -(\nu/2)V(t')$, which does not deteriorate the exponential convergence of V.

Suppose $((2\alpha^2 c_{\max}^2)/\nu)\|\mathbf{e}(t)\|^2 \geq (\nu/2)V$. By the definition of V, we have $((2\alpha^2 c_{\max}^2)/\nu)\|\mathbf{e}(t)\|^2 \geq (\nu/4)\|\tilde{\mathbf{z}}(t)\|^2$. Hence,

$$\|\tilde{\mathbf{z}}(t)\| \leq \frac{2\sqrt{2}\alpha c_{\max}}{\nu}\|\mathbf{e}(t)\|.$$

Therefore, we further have

$$\begin{aligned}\|\dot{\mathbf{e}}(t)\| &\leq 2\alpha c_{\max}\|\mathbf{e}(t)\| + (\alpha c_{\max} + l)\|\tilde{\mathbf{z}}(t)\| \\ &\leq 2\alpha c_{\max}\|\mathbf{e}(t)\| + (\alpha c_{\max} + l)\frac{2\sqrt{2}\alpha c_{\max}}{\nu}\|\mathbf{e}(t)\| \\ &\leq 2\alpha c_{\max}\left(1 + \frac{\sqrt{2}(\alpha c_{\max} + l)}{\nu}\right)\|\mathbf{e}(t)\|.\end{aligned}$$

From this, we consider the evolution of $\|\mathbf{e}(t)\|$. Denote

$$c_6 \triangleq 2\alpha c_{\max}\left(1 + \frac{\sqrt{2}(\alpha c_{\max} + l)}{\nu}\right), \quad \theta(t) = \|\mathbf{e}(t)\|$$

for short. It follows then:

$$\dot{\theta} = \frac{d(\mathbf{e}^T\mathbf{e})^{\frac{1}{2}}}{dt} = \frac{\mathbf{e}^T\dot{\mathbf{e}}}{(\mathbf{e}^T\mathbf{e})^{\frac{1}{2}}} \leq c_6\theta.$$

That is, $\|\mathbf{e}(t)\| \leq e^{c_6(t-t')}\|\mathbf{e}(t')\|$ holds for this $t > t' \geq t_k^i$ in the interval $[t_k^i,\ t_{k+1}^i)$.

For any fixed t, we let $t' = k'(t)\tau$ with $k'(t)$ being the largest integer, such that $k'\tau \leq t$. It follows then $t \geq t'(t) > t - \tau$. Since the event is not triggered at t', we have $||\mathbf{e}(t')|| \leq \sqrt{N}c_1 e^{-\beta t'}$. Then

$$\|\mathbf{e}(t) \leq e^{c_6\tau}\|\mathbf{e}(t')\| \leq e^{c_6\tau}\sqrt{N}c_1 e^{-\beta t'} \leq \sqrt{N}c_1 e^{(c_6+\beta)\tau}e^{-\beta t}$$

for any $t \in [t_k^i,\ t_{k+1}^i)$ with N the number of players. It follows then for all $t \in [t_k, t_{k+1})$:

$$\dot{V} \leq -\nu V + \varpi N\left(c_1 e^{(c_6+\beta)\tau}e^{-\beta t}\right)^2$$

where ϖ is defined as in the proof of Theorem 2. From this, we can directly solve this differential inequality and complete the proof.

Remark 4: The choice of τ is a sufficient condition in this mechanism (12). In practice, we may select a larger period τ to reduce the number of function evaluation.

Remark 5: In both event-triggered strategies, the communication only occurs when the triggering function is violated to save communication resources. For a special choice $c_0 = 0$, $c_1 > 0$, and $0 < \beta < \nu$, the Nash equilibrium seeking problem is exponentially solved, while the interval between any contiguous time instants is lower bounded by a positive constant, which rules out any Zeno behavior.

V. Simulation

This section provides the comparative simulation studies to illustrate the algorithm performance with different communication schemes and parameter settings.

Consider the Nash–Cournot game played by N industry firms in a market under an undirected cycle graph. These rival firms produce a homogeneous product, and each attempts to maximize profits by choosing how much to produce. We denote the industry output of firm i by z_i. Its production cost per unit is C_i. The price of this product is assumed to be $f(z) = D - 0.1\sum_{i=1}^N z_i + 0.45z_i$ with demand $D > 0$ and $z = \text{col}(z_1, \ldots, z_N)$. Then, for firm i, to maximize its own profit is to minimize an objective function of the form $J_i(z) = C_i z_i - z_i f(z)$. It can be easily found that these firms play a nonoperative game. In our simulations, we let $N = 5$, $C_i = 24i$, and $D = 1008$.

All assumptions in this brief can be verified with the unique Nash equilibrium as (204, 180, 156, 132, 108). To demonstrate the effectiveness of the algorithm (3), we set $\alpha = 3$ and $\tau = 0.05$ s in periodic communication, and set $c_0 = 0$, $c_1 = 10$, and $\beta = 0.35$ in both pure event-triggered and periodic event-triggered communications.

The trajectories of z_i with different discrete-time communications are presented in Fig. 1, and the corresponding error $(e_i(t) = z_i^\star - z_i, i = 1, 3, 5)$ curves with different discrete-time communications are shown in Fig. 2. It is observed that our problem is exponentially solved in all cases. We also show the communication time instants with different mechanisms in Fig. 3. The number of communications required in these simulations is 600, 326, and 236, in turn, for these three different mechanisms. We can find that the event-triggered communication indeed save communication resources in our example. These observations verify the effectiveness of our preceding designs.

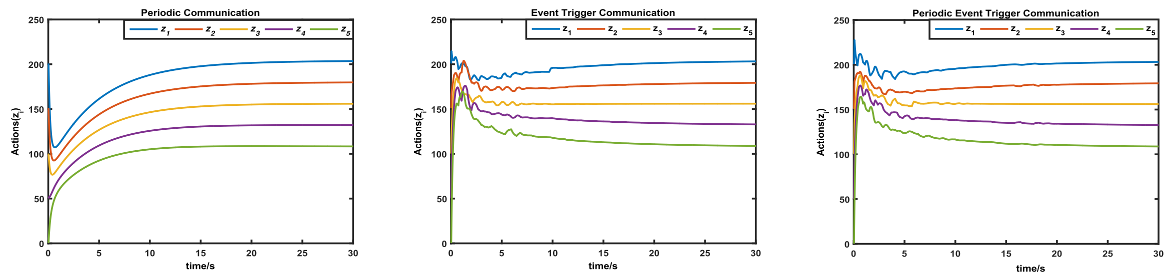

Fig. 1. Trajectories of z_i with different discrete-time communications.

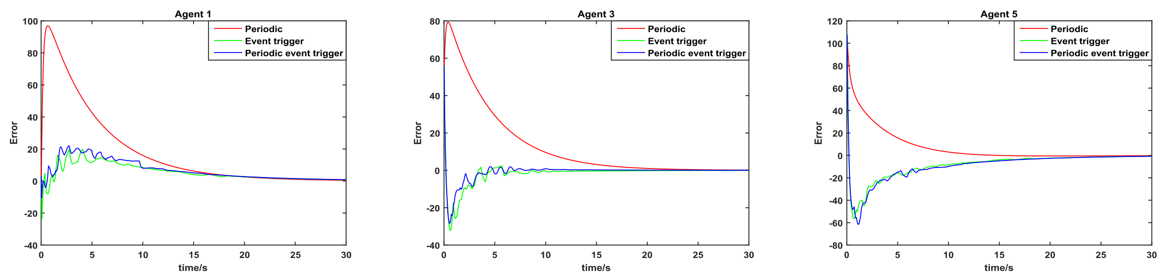

Fig. 2. Error curves with different discrete communication methods.

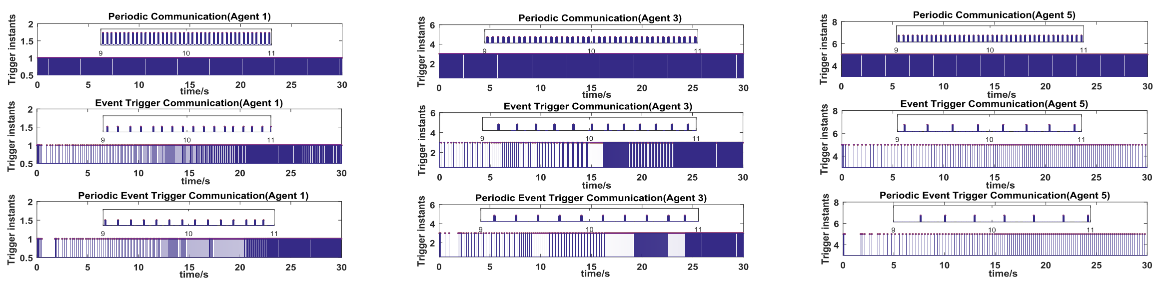

Fig. 3. Communication time instants with different mechanisms.

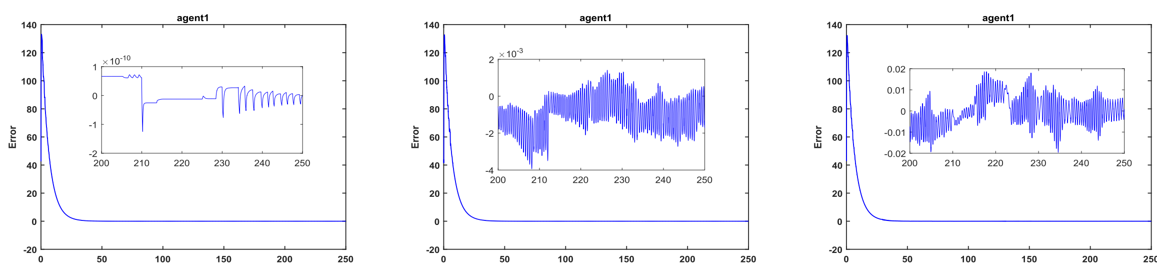

Fig. 4. Convergence errors under rule (8) with different values of c_0.

Furthermore, the convergence error ranges are exhibited for the proposed algorithm with pure event-trigger communication by designing different parameters c_0 in Fig. 4. Note that both event-triggered communications have the same property in terms of parameter c_0. When $c_0 = 0, 0.001$, and 0.01, respectively, it can be seen that the convergence error ranges are directly proportional to the parameter c_0, which is consistent with the theoretical error radius $\gamma(c_0)$. Meanwhile, the number of triggered times is 7381, 3540, and 1847, which is inversely proportional to the parameter c_0. These conclusions support and complement Theorem 2.

VI. Conclusion

In this brief, we have investigated the solvability of the distributed Nash equilibrium seeking problem by the continuous-time algorithms with different discrete-time communications. Among the novelties of our proposed algorithms, we first need to emphasize that this work does not require each agent to continuously communicate with its neighbors. Then, we highlight that three different discrete-time communication schemes have been thoroughly studied to ensure that all agents trajectories exponentially converge to the expected Nash equilibrium free from Zeno behaviors. We may extend the designs to the constrained case in future work.

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