Positive Consensus for Heterogeneous Multi-Agent Systems Over Jointly Connected Graphs

Ruonan Li, Yichen Zhang, Yutao Tang, *Member, IEEE*, and Shurong Li

***Abstract*—This letter studies the leader-following positive consensus for a group of heterogeneous multi-agent systems. The followers' dynamics are allowed to be mutually different from each other and also the leader, while the communication graph is not required to be fixed or connected. To tackle this problem, we first present an extended distributed observer for positive multi-agent systems. By constructing a novel weak common linear copositive Lyapunov function (CLCLF), we show its effectiveness by rigorous proofs over jointly connected switching networks. Then, with the aid of the presented positive observer, we propose distributed controllers for the followers to reach the expected positive consensus with disturbance rejection. Finally, a numerical simulation is provided to illustrate the effectiveness of our proposed design.**

***Index Terms*—Positive system, weak common Lyapunov function, joint connectivity, distributed control.**

I. Introduction

THE PAST two decades have witnessed intensive studies in multi-agent systems driven by their diverse and widespread applications ranging from smart grids, multi-robot systems, to sensor networks. As a fundamental problem in this area, consensus (with or without leaders) has been investigated for different kinds of multi-agent systems possessing nontrivial agent dynamics from integrators to general linear agents, and even nonlinear ones [1], [2], [3], [4], [5], [6], [7].

In this letter, we pay attention to positive multi-agent systems. Compared with standard dynamic systems, their corresponding state trajectories are always constrained in the positive orthant. This salient feature can be widely found in many practical applications, e.g., transportation networks, epidemic processes, and communication networks [8], [9], [10], [11], [12], [13]. In the seminal work [14], the consensus problem was studied from the perspective of positive systems due to the fact that single integrator are naturally positive. The authors showed that the positive consensusability problem is equivalent to a positive stabilization problem under information constraints. Since then, many extended results have been delivered in [15], [16], [17], [18], [19] by either state feedback or output feedback. Note that all the aforementioned results are derived assuming that all agents share identical dynamics, which definitely limits the application of these results in real-world scenarios.

In practice, the agents may possess different positive dynamics including system parameters and the numbers of state variables. Thus, the combination of the same feedback gain and conventional Kronecker product fails to be used for a positive consensus in the heterogeneous case. While many consensus results have been derived for standard heterogeneous linear or nonlinear multi-agent systems [3], [20], [21], [22], [23], their positive counterparts are relatively few. The main difficulty lies in how to preserve the positivity of the state trajectories while ensuring the consensus performance. In a recent attempt, we extended existing observer-based designs to positive multi-agent systems in [24]. By some careful initialization technique, a group of general linear positive multi-agent systems are driven to reach some patterned positive consensus over switching graphs. However, the time-varying interaction graph was required to be connected all time, which may be too strict in practical situations. For example, in formation control, the communication between robots does not require to happen simultaneously but instead in an order. In this case, the graph may be jointly connected but disconnected at any moment. Thus, it is interesting to consider the positive consensus problem for heterogeneous multi-agent systems over jointly connected graphs. This motivates the present study.

In this letter, we consider the leader-following positive consensus over jointly connected graphs. We assume that the agents' dynamics are high-order and can be mutually different from each other. We also consider external modelled disturbances at each follower. To solve the problem, we extend existing distributed observer-based design to positive multi-agent systems to handle the heterogeneity of agent dynamics, and convert the problem into two subproblems, i.e., positive observer design over jointly connected graphs and decentralized output regulation for positive systems. Since the graph may be disconnected at some moments, a strong common Lyapunov function may not exist for the proposed positive observer. Thus, we seek a weak common linear copositive Lyapunov function to establish the effectiveness of positive distributed observers over jointly connected graphs. Then, we combine this distributed positive observer with a local positive

Received 24 July 2024; revised 3 October 2024; accepted 27 October 2024. Date of publication 31 October 2024; date of current version 14 November 2024. This work was supported by the National Natural Science Foundation of China under Grant 62473057. Recommended by Senior Editor K. Savla. *(Corresponding author: Yutao Tang.)*

The authors are with the School of Intelligent Engineering and Automation, Beijing University of Posts and Telecommunications, Beijing 100876, China (e-mail: nanruoliy@163.com; zhangyc930@163.com; yttang@bupt.edu.cn; lishurong@bupt.edu.cn).

Digital Object Identifier 10.1109/LCSYS.2024.3489392

disturbance observer together, and solve the positive consensus problem under mild assumptions.

The contributions can be summarized as follows:

- We consider the leader-following positive consensus for heterogeneous multi-agent systems over jointly connected switching graphs. Compared with existing positive consensus results [15], [16], [17], [19], [23], [24], the agents possess mutually different dynamics and the interaction graph does not have to be fixed or always connected.
- We develop a novel distributed positive observer-based design to address the problem. By proposing a weak CLCLF, we show the effectiveness of the proposed distributed positive observers over jointly connected graphs. This together with local disturbance observers assists us to attain a leader-following positive consensus performance with disturbance rejection. To our knowledge, this might be the first attempt to solve the positive consensus problem in the same setting.

The remainder of this letter is organized as follows. Section II introduces the necessary preliminaries and provides the formulated problem. Then Section III presents our main results. After that, Section IV provides a simulation example to illustrate the efficacy of our design. Finally, we give conclusions in Section V.

Notations: Let $\mathbb{R}$ (or $\mathbb{R}_+$) be the set of (nonnegative) real numbers and $\mathbb{R}_+^{m\times n}$ the set of $m \times n$ matrices with each entry in $\mathbb{R}_+$. Let $\mathbf{1}$ (or $\mathbf{0}$) present an all-one (or all-zero) matrix or vector with proper dimensions. For $A \in \mathbb{R}^{m\times n}$, $A \geq \mathbf{0}$ means that A is (element-wise) nonnegative, i.e., $a_{ij} \geq 0$. $A > \mathbf{0}$ means that $A \geq \mathbf{0}$ and $A \neq \mathbf{0}$. We say that A is strictly positive (denoted by $A \gg \mathbf{0}$) if $a_{ij} > 0$.

II. Our Problem

Consider a group of single-input single-output positive agents consisting of one leader labeled as 0 and N followers labeled as $1, \ldots, N$. The follower $i \in \bar{\mathcal{N}} \triangleq \{1, \ldots, N\}$ is described by

$$\begin{aligned} x_i(k+1) &= A_i x_i(k) + B_i u_i(k) + E_i d_i(k) \\ y_i(k) &= C_i x_i(k), \quad k \in \mathbb{Z}_+ \triangleq \{0, 1, 2, \ldots, \} \end{aligned} \tag{1}$$

with the state $x_i \in \mathbb{R}^{n_i}$, input $u_i \in \mathbb{R}$, and output $y_i \in \mathbb{R}$. Here, the vector $d_i \in \mathbb{R}_+^{m_i}$ represents some local external disturbances at agent i generated by

$$d_i(k+1) = P_i d_i(k) \tag{2}$$

with $d_i(0) \geq \mathbf{0}$. Assume that (A_i, B_i) is positively stabilizable and (P_i, E_i) is positively detectable without loss of generality.

We aim to drive the followers to track the reference trajectory generated by the leader:

$$\begin{aligned} x_0(k+1) &= A_0 x_0(k) \\ y_0(k) &= C_0 x_0(k), \quad k \in \mathbb{Z}_+ \end{aligned} \tag{3}$$

with the internal state $x_0 \in \mathbb{R}^{n_0}$ and output $y_0 \in \mathbb{R}$. To be specific, for each agent $i \in \bar{\mathcal{N}}$, we define the tracking error as $e_i(k) = y_i(k) - y_0(k)$. Then, our objective is to ensure that $\lim_{k\to\infty} e_i(k) = 0$, while the state $x_i(k)$ is kept nonnegative for any $k \in \mathbb{Z}_+$.

To model the information flow at time $k \in \mathbb{Z}_+$, we employ a time-varying directed graph $\mathcal{G}_{\sigma(k)} = \{\mathcal{N}, \mathcal{E}_{\sigma(k)}\}$ with node set $\mathcal{N} = \{0\} \cup \bar{\mathcal{N}}$ and time-varying edge set $\mathcal{E}_{\sigma(k)} \subset \mathcal{N} \times \mathcal{N}$. Let $\bar{\mathcal{G}}_{\sigma(k)} = \{\bar{\mathcal{N}}, \bar{\mathcal{E}}_{\sigma(k)}\}$ be the induced subgraph. It represents the information flow between followers. Set $\mathcal{G}(k) = \mathcal{G}_{\sigma(k)}$ with a switching function $\sigma(k)\colon \mathbb{Z}_+ \to \mathcal{P}$ demonstrating the time-varying topologies with $\mathcal{P} = \{1, 2, \ldots, P\}$. For digraph $\mathcal{G}_p$, we remove the first column and first row from its corresponding Laplacian and obtain a matrix $H_p \in \mathbb{R}^{N\times N}$. Let $\Lambda_p = H_p - L_p$ with L_p the Laplacian of graph $\bar{\mathcal{G}}_p$. It can be verified matrix Λ_p is diagonal and its j-th element on the diagonal being 1 provided that node 0 can transmit information to node j. The time-varying matrices can be defined as $H(k) = H_{\sigma(k)}$, $L(k) = L_{\sigma(k)}$, and $\Lambda(k) = \Lambda_{\sigma(k)}$.

To proceed, we introduce some assumptions.

Assumption 1: For the leader system (3), there exists a vector $\xi \gg \mathbf{0}$ such that $\xi^\top A_0 \leq \xi^\top$.

Assumption 2: For the time-varying graph $\mathcal{G}(k)$, there exists a positive integer $T \geq 1$ such that the union digraph $\cup_{s=0}^{T-1} \mathcal{G}_{\sigma(k+s)}$ includes a spanning tree with 0 as the root node. Moreover, $\cup_{s=0}^{T-1} \bar{\mathcal{G}}_{\sigma(k+s)}$ is undirected and connected.

Remark 1: Assumption 1 specifies the leader we are interested in. Similar assumptions can be found in many papers [25], [26]. Under this assumption, the leader can be neutrally stable [10], [11] with the constants as special cases.

Remark 2: Assumption 2 is often called as joint connectivity condition [4], [27]. While most existing positive consensus designs require the communication graph to be either fixed [17], [18], [19], [28] or switching but connected all time [23], we extend them to this more general jointly connected case.

Before giving the main results, we have a few remarks on the problem formulation. First, compared with existing positive consensus results [14], [15], [16], [17], [18], [19], the agent dynamics can be mutually different from each other. Also, the interaction here allows intermittent disconnectivity rather than being always connected in the literature [15], [17], [19]. Moreover, we incorporate disturbances while the agent dynamics are often assumed to be perfectly known in existing literature.

III. Main Result

In this section, we first present a distributed positive observer over jointly connected graphs, and show its effectiveness by seeking a weak CLCLF. Then, we propose distributed controllers to solve the formulated positive consensus problem with the aid of the positive observers.

A. Positive Observer Over Jointly Connected Graphs

Motivated by existing observer-based designs [20], [21], [23], [27], we extend them to positive multi-agent systems and introduce a distributed positive observer candidate as follows:

$$v_i(k+1) = A_0 v_i(k) + \mu A_0 \sum_{j=0}^{N} a_{ij}(k)\big(v_j(k) - v_i(k)\big) \tag{4}$$

where $\mu > 0$ is a parameter to be specified later.

To investigate its performance, we let $\tilde{v}_i(k) = x_0(k) - v_i(k)$ and $\tilde{v}(k) = \mathrm{col}(\tilde{v}_1(k), \ldots, \tilde{v}_N(k))$. Then, we can obtain an error system in a compact form as follows:

$$\tilde{v}(k+1) = [(\mathbb{I}_N - \mu H(k)) \otimes A_0]\tilde{v}(k) \tag{5}$$

Due to the fact that $A_0 \geq \mathbf{0}$, matrix $(\mathbb{I}_N - \mu H(k)) \otimes A_0$ must be nonnegative when μ is sufficiently small. If we set $v_i(0) = \mathbf{0}$, the error system (5) is naturally positive. Thus, to confirm the efficacy of (4) as a positive observer for the system (3), we only have to show that this error $\tilde{v}(k)$ converges to zero when $k \to \infty$. Although this problem has been partially discussed in [24], the interaction graph is required to be connected at all time instants, which may be too strict in practical situations. Here we will remove this limitation and show its effectiveness over jointly connected graphs.

We first provide a counterexample to show that no strong CLCLF exists for this switched positive system (5) in general.

Example 1: Take $N = 3$ and $A_0 = 1$. The interaction graphs are periodically taken as $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$ with

$$H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$H_3 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Consider the following matrix:

$$\begin{aligned} &[-\mu H_1, -\mu H_2, -\mu H_3] \\ &= -\mu \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \end{aligned}$$

It can be found that the sum of the third, the sixth, and the eighth columns is $[0\ 0\ 0]^\top$. According to [10, Th. 1], we conclude that no strong CLCLF exists for this system.

With this example, a natural idea is to seek a weak CLCLF. Motivated by the fact that $\mathbf{1}^\top L_p = \mathbf{1}^\top$ under the graph assumptions, we shall show the effectiveness of the distributed positive observer (4) by constructing a novel weak CLCLF in which the interaction graphs only have to be jointly connected.

Here is a key lemma for our following analysis.

Lemma 1: Let Assumption 2 hold and $0 < \mu < \frac{1}{N}$. Then, there exists a positive constant $0 < \delta < 1$ such that

$$\mathbf{1}_N^\top (\mathbb{I}_N - \mu H_{k+NT}) \cdots (\mathbb{I}_N - \mu H_k) \le \delta^{NT} \mathbf{1}_N^\top,\ k \in \mathbb{Z}^+ \tag{6}$$

Proof: See the Appendix. ■

Remark 3: When Assumption 2 holds for $T = 1$, this lemma strictly covers the results in [24], [25] where the matrix $\Lambda(k)$ should be positive. We will show that this lemma assists us to handle more general time-varying topologies which may be disconnected at some time instants in next subsection.

Here is the first main result of this letter.

Theorem 1: Under Assumptions 1 and 2, set $w_i(0) = \mathbf{0}$ and $0 < \mu < \frac{1}{N}$. Then, $V(\tilde{v}(k)) = (\mathbf{1}^\top \otimes \xi^\top)\tilde{v}(k)$ is a weak CLCLF for system (5). Moreover, $v_i(k) \ge 0$ and $\lim_{k\to\infty} \tilde{v}(k) = \mathbf{0}$ along the trajectory of system (5).

Proof: Note that $\mathbf{1}^\top \otimes \xi^\top \gg 0$ by Assumption 1. We check the evolution of $\Delta V(\tilde{v}(k)) = V(\tilde{v}(k+1)) - V(\tilde{v}(k))$. By some mathematical manipulations, we simply have

$$\begin{aligned} \Delta V(\tilde{v}(k)) &= \left(\mathbf{1}^\top \otimes \xi^\top\right)\left[\tilde{v}(k+1) - \tilde{v}(k)\right] \\ &= -\left(\mathbf{1}^\top \otimes \xi^\top\right)(\mu H(k) \otimes A_0)\tilde{v}(k) \\ &= -\mu\left(\mathbf{1}^\top H(k)\right) \otimes \left(\xi^\top A_0\right)\tilde{v}(k) \le 0 \end{aligned}$$

This implies that $V(\tilde{v}(k))$ is indeed a weak CLCLF for (5).

While the positivity of $v_i(k)$ is trivial, we show the convergence of $\tilde{v}_i(k)$ towards 0 as k goes to ∞. Note that

$$\begin{aligned} V(\tilde{v}(k+NT)) &= \left(\mathbf{1}^\top \otimes \xi^\top\right)[(\mathbb{I}_N - \mu H(k+NT)) \\ &\quad \cdots (\mathbb{I}_N - \mu H(k)) \otimes \left(A_0^{NT}\right)]\tilde{v}(k) \\ &\le \delta^{NT}\left(\mathbf{1}^\top \otimes \xi^\top\right)\tilde{v}(k) = \delta^{NT} V(\tilde{v}(k)) \end{aligned}$$

where we have used the property in Lemma 1. Repeating the above inequality, we have $V(\tilde{v}(k)) \le \delta^{\lfloor \frac{k}{NT} \rfloor} V(\tilde{v}(0))$. This implies that $\tilde{v}(k)$ goes to zero as $k \to \infty$. ■

Remark 4: Theorem 1 shows the effectiveness of (4) as a distributed positive observer for the leader (3) over jointly connect graphs. It can be taken as an extension of the invariance-like theorems in [29] but for positive switching systems, which strictly covers similar conclusions in [24] without requiring the interaction graph to be always connected.

B. Leader-Following Positive Consensus

We shall address our positive consensus problem by virtue of the established distributed positive observer (4). To this end, another assumption on the agent dynamics is introduced.

Assumption 3: For any $i \in \mathcal{N}$, there exist constant matrices $X_i \in \mathbb{R}_+^{n_i \times n_0}$, $U_i \in \mathbb{R}_+^{1\times n_0}$, and $M_i \in \mathbb{R}_+^{1\times m}$ such that

$$\begin{aligned} &A_i X_i + B_i U_i - X_i A_0 = \mathbf{0}, \quad C_i X_i - C_0 = \mathbf{0} \\ &E_i - B_i M_i = \mathbf{0} \end{aligned} \tag{7}$$

Remark 5: This assumption summarizes the requirement on the agents' dynamics to reach the expected positive consensus with disturbance rejection. It can be understood as the regulator equations in the terminology of output regulation theory (see [30, Sec. 1.1] and [31, Sec. 4.2]). Similar assumptions have been widely used in distributed designs for both standard and positive multi-agent systems [19], [20], [21], [24], [27].

Since the followers are subject to unknown external disturbances generated by (2), we utilize a reduced-order positive observer to estimate $d_i(k)$ as follows:

$$\begin{aligned} \zeta_i(k+1) &= (P_i - L_i E_i)\hat{d}_i(k) - L_i A_i x(k) - L_i B_i u_i(k) \\ \hat{d}_i(k) &= \zeta_i(k) + L_i x_i(k) \end{aligned} \tag{8}$$

where $\hat{d}_i(k)$ is the estimate of $d_i(k)$ and $L_i \in \mathbb{R}^{1\times n_i}$ is a gain matrix such that $P_i - L_i E_i$ is nonnegative and Schur.

Combining (4) and (8) together, we present the final distributed controller for agent i to solve the leader-following positive consensus problem of the following form:

$$\begin{aligned} u_i(k) &= -K_{1i} x_i(k) + K_{2i} v_i(k) + K_{3i} \hat{d}_i(k) \\ v_i(k+1) &= A_0 v_i(k) + \mu A_0 \sum\nolimits_{j=0}^{N} a_{ij}(k)\big(v_j(k) - v_i(k)\big) \\ \zeta_i(k+1) &= (P_i - L_i E_i)\hat{d}_i(k) - L_i A_i x(k) - L_i B_i u_i(k) \\ \hat{d}_i(k) &= \zeta_i(k) + L_i x_i(k) \end{aligned} \tag{9}$$

where $\zeta_i(0) = -L_i x_i(0)$, $v_i(0) = \mathbf{0}$, and the matrices K_{1i}, K_{2i}, and K_{3i} are gain matrices to be determined by us.

Here is the second main theorem to solve the formulated positive consensus for multi-agent systems (1) and (3).

Theorem 2: Suppose that Assumptions 1-3 hold. Then, the leader-following positive consensus problem for heterogeneous multi-agent systems (1) and (3) can be solved by a distributed controller of the form (9) irrespective of any disturbance generated by (2). That is, for any $x_i(0)$, $x_0(0)$, $d_i(0) \ge \mathbf{0}$, along the trajectory of the composite system (1)–(3) and (9), $x_i(k) \ge \mathbf{0}$ and $\lim_{k\to\infty} e_i(k) = 0$.

Proof: First, we let $0 < \mu < \frac{1}{N}$ and choose a matrix $K_{1i} \ge \mathbf{0}$ such that $A_i - B_i K_{1i}$ is nonnegative and Schur. This is guaranteed by [32, Lemma 6]. Let $K_{2i} = U_i + K_{1i} X_i$ and $K_{3i} =$

$-M_i$ with U_i, X_i, M_i the solution to (7). By Assumption 3, $K_{2i} \geq \mathbf{0}$ and $K_{3i} \leq \mathbf{0}$.

We first show that the state $x_i(k) \geq \mathbf{0}$. For this purpose, we put (1)–(3) and controller (9) together. It follows that

$$\begin{aligned} x_i(k+1) &= \hat{A}_i x_i(k) + B_i K_{2i} v_i(k) + E_i d_i(k) + B_i K_{3i} \hat{d}_i(k) \\ v_i(k+1) &= A_0 v_i(k) + \mu A_0 \sum\nolimits_{j=0}^{N} a_{ij}(k)\big(v_j(k) - v_i(k)\big) \\ \zeta_i(k+1) &= (P_i - L_i E_i)\hat{d}_i(k) - L_i A_i x(k) - L_i B_i u_i(k) \end{aligned}$$

where $\hat{A}_i = A_i - B_i K_{1i}$. Define the error $\tilde{d}_i(k)$ as $d_i(k) - \hat{d}_i(k)$. The above system can be further written as:

$$\begin{aligned} x_i(k+1) &= \hat{A}_i x_i(k) + B_i K_{2i} v_i(k) - B_i K_{3i} \tilde{d}_i(k) \\ v_i(k+1) &= A_0 v_i(k) + \mu A_0 \sum\nolimits_{j=0}^{N} a_{ij}(k)\big(v_j(k) - v_i(k)\big) \\ \tilde{d}_i(k+1) &= (P_i - L_i E_i)\tilde{d}_i(k) \end{aligned} \tag{10}$$

Since $\hat{d}_i(0) = \mathbf{0}$ by our initialization, we have $\tilde{d}_i(0) \geq \mathbf{0}$. Combined with the fact that matrix $P_i - L_i E_i$ is nonnegative and Schur, it implies that $\tilde{d}(k) \geq \mathbf{0}$ and $\lim_{k\to\infty} \tilde{d}(k) = \mathbf{0}$. From Theorem 1, we know that $v_i(k) \geq \mathbf{0}$. Thus, for i-th subsystem, we view $B_i K_{2i} v_i(k) - B_i K_{3i} \tilde{d}_i(k)$ as an input and conclude that $x_i(k) \geq \mathbf{0}$.

Next, we show that $e_i(k)$ converges to 0 when k approaches infinity. We let $\tilde{x}_i(k) = x_i(k) - X_i x_0(k)$ and have

$$\begin{aligned} \tilde{x}_i(k+1) &= \hat{A}_i \tilde{x}_i(k) - B_i\Big(K_{2i}\tilde{v}_i(k) + K_{3i}\tilde{d}_i(k)\Big) \\ \tilde{v}(k+1) &= \big[\big(\mathbb{I}_N - \mu H_{\sigma(k)}\big) \otimes A_0\big]\tilde{v}(k) \\ \tilde{d}_i(k+1) &= (P_i - L_i E_i)\tilde{d}_i(k) \end{aligned} \tag{11}$$

The choice of gain matrices ensures that $\hat{A}_i$ is Schur. Consequently, the $\tilde{x}_i$-subsystem is input-to-state stable with respect to state $\tilde{x}_i$ and input $K_{2i}\tilde{v}_i(k) + K_{3i}\tilde{d}_i(k)$ in the sense of [33, Definition 3.1]. Meanwhile, by Theorem 1 and the choice of L_i, $\tilde{v}_i(k)$ and $\tilde{d}_i(k)$ both approach to zero as $k \to \infty$. Hence, these conditions allow us to conclude that $\lim_{k\to\infty} \tilde{x}_i(k) = \mathbf{0}$, which immediately yields $\lim_{k\to\infty} e_i(k) = 0$. ■

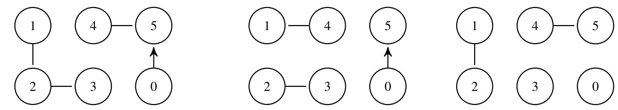
In particular, when the local disturbance $d_i(k)$ vanishes, a simplified distributed controller can be given as follows:

$$\begin{aligned} u_i(k) &= -K_{1i} x_i(k) + K_{2i} v_i(k) \\ v_i(k+1) &= A_0 v_i(k) + \mu A_0 \sum\nolimits_{j=0}^{N} a_{ij}(k)\big(v_j(k) - v_i(k)\big) \end{aligned} \tag{12}$$

Following the same arguments in Theorem 2, we can immediately show the effectiveness of this simplified controller to solve our problem under the same assumptions.

Remark 6: Compared with positive consensus results in [15], [16], [17], [18], [23], [24], the agents are not required to be homogeneous and exactly known. The disturbed agent dynamics are high-order and can be mutually different from each other.

Remark 7: While the graph was required to be always connected in some existing works [15], [16], [17], [19], [23], [24], [28], we remove this limitation and guarantee a leader-following positive consensus for high-order multi-agent systems over jointly connected graphs.



(a) $\mathcal{G}_1$ (b) $\mathcal{G}_2$ (c) $\mathcal{G}_3$

Fig. 1. The interaction graphs in our example.

IV. SIMULATION

In this section, we illustrate the effectiveness of our proposed designs by a numerical example.

We consider a multi-agent system consisting of six agents with system matrices as follows:

$$\begin{aligned} A_0 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, C_0 = \begin{bmatrix} 1 & 1 & 0.5 \end{bmatrix} \\ A_1 = A_2 &= \begin{bmatrix} 0.1 & 0.2 & 0.3 \\ 0.1 & 0.3 & 0.2 \\ 0.2 & 0.25 & 0.12 \end{bmatrix}, B_1 = B_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \\ C_1 = C_2 &= \begin{bmatrix} 1 & 0 & 2 \end{bmatrix}, E_1 = E_2 = \begin{bmatrix} 0.3 & 0.3 & 0.6 \end{bmatrix}^\top \\ A_3 = A_4 &= \begin{bmatrix} 0.3 & 0.4 \\ 0.3 & 0.23 \end{bmatrix}, \quad B_3 = B_4 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ C_3 = C_4 &= \begin{bmatrix} 0 & 2 \end{bmatrix}, E_3 = E_4 = \begin{bmatrix} 1 & 2 \end{bmatrix}^\top \\ A_5 &= 0.5, \quad B_5 = 1, \quad C_5 = 1, \quad E_5 = \begin{bmatrix} 1 & 0.9 \end{bmatrix} \\ P_1 &= P_2 = 1, P_3 = P_4 = 1.2, P_5 = \begin{bmatrix} 1 & 0.2 \\ 0.3 & 1.2 \end{bmatrix} \end{aligned}$$

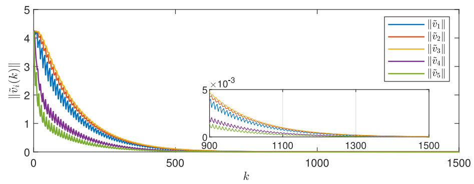
By Lemma 1 in [24], all agents' dynamics are positive. One can also verify Assumption 1 with $\xi = [2 \;\; 1 \;\; 1]^\top$. Suppose the interaction graph is switching in the order $\{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\}$ in Fig. 1 every 3 steps. Assumption 2 holds for $T = 3$. Then, numerically solving equations (7) yields

$$\begin{aligned} X_1 = X_2 &= \begin{bmatrix} 0.2853 & 0.2404 & 0.1874 \\ 0.2772 & 0.2422 & 0.1737 \\ 0.3574 & 0.3798 & 0.1563 \end{bmatrix} \\ U_1 = U_2 &= \begin{bmatrix} 0.0941 & 0.0010 & 0.1401 \end{bmatrix} \\ X_3 = X_4 &= \begin{bmatrix} 0.4618 & 0.3697 & 0.3230 \\ 0.5000 & 0.5000 & 0.2500 \end{bmatrix} \\ U_3 = U_4 &= \begin{bmatrix} 0.1232 & 0.0120 & 0.1728 \end{bmatrix} \\ X_5 &= \begin{bmatrix} 1.0000 & 1.0000 & 0.5000 \end{bmatrix}^\top \\ U_5 &= \begin{bmatrix} 0.5000 & 0 & 0.7500 \end{bmatrix}^\top \\ M_1 = M_2 &= 0.3, \quad M_3 = M_4 = 1, \quad M_5 = \begin{bmatrix} 1 & 0.9 \end{bmatrix}^\top \end{aligned}$$

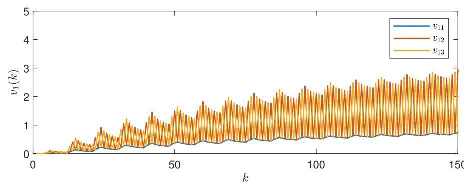
Hence Assumption 3 holds. As a result, we can resort to Theorem 2 to address our problem using controller (9).

For simulations, we let $\mu = 0.16$, $v_i(0) = \mathbf{0}$, and choose $L_1 = L_2 = \begin{bmatrix} 1 & 1 & 0.5 \end{bmatrix}$, $L_3 = L_4 = \begin{bmatrix} 0.3 & 0.2 \end{bmatrix}$, and $L_5 = \begin{bmatrix} 0.8 & 1 \end{bmatrix}^\top$. Moreover, the initial conditions of $x(k)$ are randomly generated between 0 and 7. The initial conditions of $d(k)$ are randomly generated between 0 and 7. The performance of our distributed positive observer is depicted in Fig. 2. It can be found that $\lim_{k\to\infty} \|\tilde{v}_i(k)\| = 0$. We also list the component-wise profile of $v_1(k)$ as an example to illustrate the positivity of the estimates.

Next, we choose $K_{11} = K_{12} = \begin{bmatrix} 0.1 & 0.1 & 0.06 \end{bmatrix}$, $K_{31} = K_{32} = -0.3$, $K_{21} = K_{22} = \begin{bmatrix} 0.1718 & 0.0721 & 0.1856 \end{bmatrix}$, $K_{13} =$

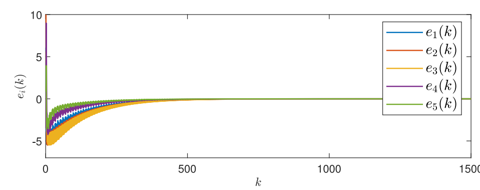


(a) $\|\tilde{v}_i(k)\|$

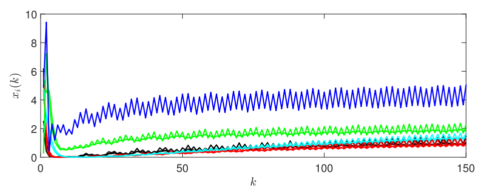


(b) $v_1(k)$

Fig. 2. Profile of $\|\tilde{v}_i(k)\|$ and $v_1(k)$ by distributed observer (4).



(a) $e_i(k)$



(b) $x_i(k)$

Fig. 3. Profile of $e_i(k)$ and $x_i(k)$ under controller (9).

$K_{14} = \begin{bmatrix}0.15 & 0.1\end{bmatrix}$, $K_{33} = K_{34} = -1$, $K_{23} = K_{24} = \begin{bmatrix}0.2425 & 0.1175 & 0.2463\end{bmatrix}$, $K_{15} = 0.2$, $K_{35} = -\begin{bmatrix}1 & 0.9\end{bmatrix}$, and $K_{25} = \begin{bmatrix}0.7000 & 0.2000 & 0.8500\end{bmatrix}$. Fig. 3 shows the profiles of the tracking error $e_i(k)$ and state trajectories of all agents (by different colors) under the controller (9). It can be found that $x_i(k) \geq \mathbf{0}$ and the leader-following consensus is finally reached. These observations verify the efficacy of our proposed design in addressing the positive consensus problem over jointly connected graphs.

V. Conclusion

A leader-following positive consensus has been investigated for heterogeneous multi-agent systems over jointly connected topologies. By presenting a novel distributed positive observer-based design, we have solved the formulated positive consensus problem over jointly connected graphs. Future research directions include robustness of our designs against unmodelled disturbances and communication delays.

Appendix

Proof of Lemma 1

Before proving this lemma, we can trivially verify that $[\mathbf{1}_N^\top(\mathbb{I}_N - \mu H_p)]_i \leq 1$ for any $p \in \mathcal{P}$ and $i \in \bar{\mathcal{N}}$. To show the strictness of the inequality (6) during a long enough time interval, we introduce the following two sets for each $p \in \mathcal{P}$:

$$\mathcal{I}_p^1 \triangleq \{i \in \bar{\mathcal{N}}\colon \left[\mathbf{1}_N^\top(\mathbb{I}_N - \mu H_p)\right]_i < 1\}$$
$$\mathcal{I}_p^2 \triangleq \{j \in \bar{\mathcal{N}}\colon \mathbf{1}_N^\top(\mathbb{I}_N - \mu H_p)]_j = 1\}$$

For an ordered sequence of graphs $\{\mathcal{G}(k)\}$, we further define

$$\mathcal{N}^1(k,\, k+l) \triangleq \{i \in \bar{\mathcal{N}}\colon [\mathbf{1}_N^\top(\mathbb{I}_N - \mu H(k+l)) \cdots (\mathbb{I}_N - \mu H(k)]_i < 1\}$$
$$\mathcal{N}^2(k,\, k+l) \triangleq \{j \in \bar{\mathcal{N}}\colon [\mathbf{1}_N^\top(\mathbb{I}_N - \mu H(k+l)) \cdots (\mathbb{I}_N - \mu H(k)]_j = 1\}$$

for any two nonnegative integers k and l. Apparently, $\mathcal{N}^1(k,\, k) = \mathcal{I}^1_{\sigma(k)}$. With the aforementioned notations, we only need to prove that

$$\mathcal{N}^1(k,\, k+NT) = \bar{\mathcal{N}}, \quad \mathcal{N}^2(k,\, k+NT) = \emptyset \tag{13}$$

for any $k \in \mathbb{Z}^+$. In fact, since the set $\{\mathcal{G}_p\}$ is finite, we can simply set δ^{NT} to be the minimal $[\mathbf{1}_N^\top(\mathbb{I}_N - \mu H_{p_{NT}}) \cdots (\mathbb{I}_N - \mu H_{p_1})]_i$ with $p_1, \ldots, p_{NT} \in \mathcal{P}$ to meet the inequality (6).

To prove the relationship (13), we first claim that $\mathcal{N}^1(k,\, k+l) \subset \mathcal{N}^1(k,\, k+l+1)$ for any nonnegative integers k and l. We prove this by an induction of l. When $l = 0$, $\mathcal{N}^1(k,\, k) = \{i\colon b_i(k) \neq 0\}$ with $b_i(\ell)$ the i-th element in the diagonal of matrix $\Lambda(\ell)$. Preserving the generality of our analysis, we suppose that $\mathcal{N}^1(k,\, k+1) \neq \emptyset$ and $\iota \in \mathcal{N}^1(k,\, k+1)$. Then, $[\mathbf{1}_N^\top(\mathbb{I}_N - \mu H(k+1))]_\iota = 1 - \mu$ and $[\mathbf{1}_N^\top(\mathbb{I}_N - \mu H(k+1))]_{i \neq \iota} \leq 1$. Let us consider the set $\mathcal{N}^1(k,\, k+1)$ determined by the vector $\mathbf{1}_N^\top(\mathbb{I}_N - \mu H(k+1))(\mathbb{I}_N - \mu H(k))$. Its ι-th element satisfies the following inequality:

$$\begin{aligned}\Big[\mathbf{1}_N^\top(\mathbb{I}_N &- \mu H(k+1))(\mathbb{I}_N - \mu H(k))\Big]_\iota \\ &\leq 1 - \mu + \mu^2 l_{\iota\iota}(k) - \mu b_\iota(k) \\ &\leq 1 - \mu + \mu \frac{N-1}{N} < 1\end{aligned}$$

in which $l_{\iota\iota}(\ell)$ is the ι-th entry in the diagonal of the matrix $L(\ell)$. Hence, we have $\iota \in \mathcal{N}^1(k,\, k+1)$. Suppose that $\mathcal{N}^1(k,\, k+\bar{\ell}) \subset \mathcal{N}^1(k,\, k+\bar{\ell}+1)$ for some $\bar{\ell}$. We consider the set $\mathcal{N}^1(k,\, k+\bar{\ell}+2)$. We temporally write the vector $\mathbf{1}_N^\top(\mathbb{I}_N - \mu H(k+\bar{\ell}+1)) \cdots (\mathbb{I}_N - \mu H(k))$ in an element-wise form as $\eta = [\,\eta_1 \;\cdots\; \eta_N\,]^\top$. Then, for any $\iota \in \mathcal{N}^1(k,\, k+\bar{\ell}+1)$, $\eta_\iota < 1$ and $\eta_{i \neq \iota} \leq 1$. Then, the ι-th element of the vector $\mathbf{1}_N^\top(\mathbb{I}_N - \mu H(k+\bar{\ell}+2)) \cdots (\mathbb{I}_N - \mu H(k))$ is given by

$$\begin{aligned}\Big[\mathbf{1}_N^\top(\mathbb{I}_N &- \mu H(k+\bar{\ell}+2)) \cdots (\mathbb{I}_N - \mu H(k))\Big]_\iota \\ &\leq \Big[\mathbf{1}_N^\top(\mathbb{I}_N - \mu H(k+\bar{\ell}+1)) \cdots (\mathbb{I}_N - \mu H(k))\Big]_\iota \\ &\leq \eta_\iota < 1\end{aligned}$$

That is, $\iota \in \mathcal{N}^1(k,\, k+\bar{\ell}+2)$. Overall, our claim is proved.

Next, we show that the set inclusion relationship $\mathcal{N}^1(k,\, k+l) \subset \mathcal{N}^1(k,\, k+l+T)$ is strict when $\mathcal{N}^2(k,\, k+l) \neq \emptyset$. We start from the special case and then consider the general case.

i) *Special case when* $T = 1$: For this case, we prove it by seeking a contradiction. Suppose that $\mathcal{N}^2(k,\, k+\bar{\ell}) \neq \emptyset$ for some $\bar{\ell} > 0$, and we have $\mathcal{N}^1(k,\, k+\bar{\ell}) = \mathcal{N}^1(k,\, k+\bar{\ell}+1)$. By the graph connectivity assumption, there must be some node $\iota_2 \in \mathcal{N}^2(k,\, k+\bar{\ell})$ connected to another node $\iota_1 \in \mathcal{N}^1(k,\, k+\bar{\ell})$ through an edge in $\mathcal{G}(k+\bar{\ell}+1)$. Again, we temporally write the vector $\mathbf{1}_N^\top(\mathbb{I}_N-\mu H(k+\bar{\ell}))\cdots(\mathbb{I}_N-\mu H(k))$ in an element-wise form as $\eta=[\,\eta_1\ \cdots\ \eta_N\,]^\top$. Then, we have $\eta_{\iota_2}=1$, $\eta_{\iota_1}<1$, and $\eta_{i\notin\{\iota_1,\,\iota_2\}}\leq 1$. Since the interaction graph is arbitrarily switching, we consider the ι_2-th element of vector $\mathbf{1}_N^\top(\mathbb{I}_N-\mu H(k))\cdots(\mathbb{I}_N-\mu H(k+\bar{\ell}+1))$. It is given by

$$\begin{aligned}\Big[\eta^\top(\mathbb{I}_N-\mu H(k+\bar{\ell}+1))\Big]_{\iota_2}&\\ \leq 1-\mu(1-\eta_{\iota_1})l_{\iota_1\iota_2}(k+\bar{\ell}+1)-\mu b_{\iota_2}(k+\bar{\ell}+1)&\\ <1 &\qquad (14)\end{aligned}$$

where $l_{\iota_1\iota_2}(k+\bar{\ell}+1)$ represents the $(\iota_1,\,\iota_2)$-th element of the matrix $L(k+\bar{\ell}+1)$. It is unity by assumptions. From here, one can obtain that $\iota_2\in\mathcal{N}^1(k,\,k+\bar{\ell}+1)=\mathcal{N}^1(k,\,k+\bar{\ell})$, which is a contradiction. Hence, the set inclusion relationship is strict when $\mathcal{N}^2(k,\,k+l)\neq\emptyset$.

ii) *General case when* $T>1$: In this case, $\mathcal{N}^1(k,\,k+l)\subset\mathcal{N}^1(k,\,k+l+1)$ may be not strict even when $\mathcal{N}^2(k,\,k+l)\neq\emptyset$ for any $k\in\mathbb{Z}^+$. Nevertheless, we claim that if $\mathcal{N}^2(k,\,k+l)\neq\emptyset$ for any $k,\ l\in\mathbb{Z}^+$, the set inclusion relationship $\mathcal{N}^1(k,\,k+l+\ell)\subset\mathcal{N}^1(k,\,k+l+\ell+1)$ must be strict for some $0\leq\ell\leq T-1$. If so, we can trivially obtain that $\mathcal{N}^1(k,\,k+l)\subset\mathcal{N}^1(k,\,k+l+T)$ is strict when $\mathcal{N}^2(k,\,k+l)\neq\emptyset$. To prove this claim, we assume that $\mathcal{N}^1(k,\,k+l)=\cdots=\mathcal{N}^1(k,\,k+l+T)$, and seek a contradiction. Since $\mathcal{N}^2(k,\,k+l)\neq\emptyset$, by the graph assumption, there must be two nodes $\iota_1\in\mathcal{N}^1(k,\,k+l)$ and $\iota_2\in\mathcal{N}_2(k,\,k+l)$ connected by an edge at some time instant $k+l+\ell$ with $0\leq\ell\leq T-1$. At this time instant, one can follow the same arguments for inequality (14) and obtain that $\iota_2\in\mathcal{N}^1(k,\,k+l+T)=\mathcal{N}^1(k,\,k+l)$. This is a contradiction.

Overall, we combine the two properties of the sets $\{\mathcal{N}^1(k,\,k),\,\ldots,\,\mathcal{N}^1(k,\,k+NT)\}$ together, and conclude that $\mathcal{N}^1(k,\,k+NT)=\mathcal{N}$ and $\mathcal{N}^2(k,\,k+NT)=\emptyset$ for any $k\in\mathbb{Z}^+$. The proof is thus complete.

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