Nash Equilibrium Seeking for High-Order Multiagent Systems With Unknown Dynamics

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***Abstract*—In this article, we consider a Nash equilibrium seeking problem for a class of high-order multiagent systems with unknown dynamics. Different from existing results for single integrators, we aim to steer the outputs of this class of uncertain high-order agents to the Nash equilibrium of some noncooperative game in a distributed manner. To overcome the difficulties brought by the high-order structure, unknown nonlinearities, and the regulation requirement, we first introduce a virtual player for each agent and solve an auxiliary noncooperative game for them. Then, we develop a distributed adaptive protocol by embedding this auxiliary game dynamics into some proper tracking controller for the original agent to resolve this problem. We also discuss the parameter convergence issue under certain persistence of excitation conditions. The efficacy of our algorithms is verified by numerical examples.**

***Index Terms*—Adaptive control, embedded design, Nash equilibrium, unknown dynamics.**

I. Introduction

NASH equilibrium computation is one of the most fundamental problems in noncooperative game theory and has been studied for many years [1]. With the wide applications of multirobot networks and Big Data technologies, considerable efforts have been made in designing noncooperative games for the engineering multiagent systems to ensure that the final Nash equilibrium is desirable to meet some system-level constraints and objectives [2], [3], [4], [5]. In these multiagent games, each player/agent only has access to its own and neighboring information for some local computation. Thus, how to develop distributed rules to seek or learn a Nash equilibrium has become a hot topic over the past few years. Many important results have been obtained under various circumstances, see [6], [7], [8], [9], [10], [11], [12], [13] and references therein.

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In most continuous-time Nash equilibrium seeking results, the players are often assumed to be single integrators from the perspective of computation. However, in many practical applications, the decision process for each player might have nontrivial internal dynamics and be subject to various certainties or disturbances. These issues from the agent dynamics will inevitably affect and even deteriorate the performance of existing Nash equilibrium seeking algorithms specially designed for single integrators. Thus, it is crucial for us to take the possible uncertain high-order agent dynamics into consideration to complete the Nash equilibrium seeking tasks. Although efforts have been made by some authors for full-information circumstances, e.g., [14], [15], [16], [17], there are very few works on the solvability of distributed Nash equilibrium seeking problem under the partial information scenario for high-order multiagent systems.

Recently, some interesting attempts have been made along this line for nonsingle-integrator multiagent systems to distributedly reach a steady-state related to Nash equilibria of some noncooperative games [18], [19], [20], [21], [22], [23]. Different from the classical works under the name of the differential game [1], the cost of each agent in these works is described by a function depending upon the agents' current outputs rather than a functional involving some integral-type running cost. This has been greatly motivated by the wide applications of engineering multiagent system implementing designed distributed algorithms to complete many challenging tasks, e.g., formation control and area coverage [2], [24].

Several special classes of high-order multiagent systems have been discussed in the aforementioned works. In [18], the gradient-play rules were extended to solve the Nash equilibrium seeking problem for (multiple) integrators with disturbance rejection. In [19], Bianchi and Grammatico considered a generalized Nash equilibrium seeking problem with coupling constraints and solved it for double-integrator multiagent systems. Bounded controls were further developed in [20] to overcome the input saturation of single or double-integrator agents. In a very recent work [21], the agent dynamics are even allowed to be general linear systems subject to small model uncertainties and external disturbances via some internal model-based designs when the cost functions are quadratic. With regard to special types of aggregative games, more general agent dynamics have also been explored. For example, passive nonlinear second-order agents were considered in [25] by a proportional integral feedback algorithm to reach the expected Cournot–Nash equilibrium. Without assuming the exact knowledge of agent dynamics, Deng and Liang [22] and Zhang et al. [23] further took parameter uncertainties into consideration and developed effective distributed rules to drive the outputs of agents in the Euler–Lagrange form and output feedback form with unity relative degree to reach the Nash equilibrium of some aggregative games.

In this work, we consider a noncooperative game played by a class of nonlinear high-order multiagent systems with unknown dynamics. More specifically, we focus on the case when the unknown time-varying dynamics can be linearly parameterized. As discussed in [18], [19], [21], [22], and [23], we aim to drive the steady-state output of this high-order nonlinear multiagent system to reach a Nash equilibrium specified by the given noncooperative game irrespective of the uncertainties in the agent dynamics. The contribution of this article is at least twofold. On the one hand, we solve a Nash equilibrium seeking problem for a class of high-order nonlinear multiagent systems subject to unknown dynamics. When such unknown dynamics vanishes, the agent dynamics can include both single and multiple integrators as special cases. Thus, this work can be taken as an adaptive high-order extension of the results obtained in [9], [11], and [18]. On the other hand, we explore the possibility of an embedded control approach to solve such a Nash equilibrium seeking problem for high-order multiagent systems. Although initiated for distributed optimization in [26], this embedded design is proven to be able to substantially reduce the design complexities and facilitate us to solve the Nash equilibrium seeking problem for high-order agents in a modular way. The rest of this article is organized as follows. Some preliminaries are provided in Section II. Problem formulation is presented in Section III. Then, the main results are given in Section IV along with both solvability analysis and parameter convergence. Following that, several examples are provided to illustrate the effectiveness of our algorithms in Section V. Finally, Section VI concludes this article.

II. Preliminary

In this section, we present some preliminaries of our notations and graph theory for the following analysis. We use standard notations. Let $\mathbb{R}^N$ be the N-dimensional Euclidean space and $\mathbb{R}^{N_1\times N_2}$ be the set of all $N_1 \times N_2$ matrices. $\mathbf{1}_N$ (or $\mathbf{0}_N$) denotes an N-dimensional all-one (or all-zero) column vector and $\mathbf{1}_{N_1\times N_2}$ (or $\mathbf{0}_{N_1\times N_2}$) all-one (or all-zero) matrix. $\text{diag}\{b_1, \ldots, b_N\}$ denotes an $N \times N$ diagonal matrix with diagonal elements b_i with $i = 1, \ldots, N$. $\text{blockdiag}(A_1, \ldots, A_N)$ denotes a block diagonal matrix with diagonal elements A_i with $i = 1, \ldots, N$. $\text{col}(a_1, \ldots, a_N) = [a_1^\intercal, \ldots, a_N^\intercal]^\intercal$ for column vectors a_i with $i = 1, \ldots, N$. For a vector x and a matrix A, $||x||$ denotes the Euclidean norm and $||A||$ the spectral norm. Let $M_1 = \frac{1}{\sqrt{N}}\mathbf{1}_N$ and $M_2 \in \mathbb{R}^{N\times(N-1)}$ be the matrix satisfying $M_2^\intercal M_1 = \mathbf{0}_{N-1}$, $M_2^\intercal M_2 = I_{N-1}$ and $M_2 M_2^\intercal = I_N - M_1 M_1^\intercal$. We may omit the subscript when it is self-evident. A vector-valued function $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ is said to be ω-strongly monotone, if for any $\zeta_1, \zeta_2 \in \mathbb{R}^m$, we have $(\zeta_1 - \zeta_2)^\intercal[\Phi(\zeta_1) - \Phi(\zeta_2)] \geq \omega\|\zeta_1 - \zeta_2\|^2$. Function $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ is said to be ϑ-Lipschitz, if for any $\zeta_1, \zeta_2 \in \mathbb{R}^m$, it holds that $\|\Phi(\zeta_1) - \Phi(\zeta_2)\| \leq \vartheta\|\zeta_1 - \zeta_2\|$.

A weighted directed graph (or digraph) $\mathscr{G} = (\mathscr{N}, \mathscr{E}, \mathscr{A})$ is defined as follows, where $\mathscr{N} = \{1, \ldots, N\}$ is the set of nodes, $\mathscr{E} \subset \mathscr{N} \times \mathscr{N}$ is the set of edges, and $\mathscr{A} \in \mathbb{R}^{N\times N}$ is a weighted adjacency matrix. $(i, j) \in \mathscr{E}$ denotes an edge leaving from node i and entering node j. The weighted adjacency matrix is described by $\mathscr{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$, where $a_{ii} = 0$ and $a_{ij} \geq 0$ ($a_{ij} > 0$ if and only if there is an edge from agent j to agent i). A path in graph $\mathscr{G}$ is an alternating sequence $i_1 e_1 i_2 e_2 \cdots e_{k-1} i_k$ of nodes i_l and edges $e_m = (i_m, i_{m+1}) \in \mathscr{E}$ for $l = 1, 2, \ldots, k$. The neighbor set of agent i is defined as $\mathscr{N}_i = \{j : (j, i) \in \mathscr{E}\}$ for $i \in \mathscr{N}$. If there is a directed path between any two nodes, then the digraph is said to be strongly connected.

The in-degree and out-degree of node i are defined by $d_i^{\text{in}} = \sum_{j=1}^N a_{ij}$ and $d_i^{\text{out}} = \sum_{j=1}^N a_{ji}$. A digraph is weight-balanced if $d_i^{\text{in}} = d_i^{\text{out}}$ holds for any $i = 1, \ldots, N$. The Laplacian matrix of $\mathscr{G}$ is defined as $L \triangleq D^{\text{in}} - \mathscr{A}$ with $D^{\text{in}} = \text{diag}(d_1^{\text{in}}, \ldots, d_N^{\text{in}})$. Note that $L\mathbf{1}_N = \mathbf{0}_N$ for any digraph. When it is weight-balanced, we have $\mathbf{1}_N^\intercal L = \mathbf{0}_N^\intercal$ and the matrix $\text{Sym}(L) \triangleq \frac{L+L^\intercal}{2}$ is positive semidefinite. Then, we can order the eigenvalues of $\text{Sym}(L)$ as $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$. A weight-balanced digraph is strongly connected if and only if $\lambda_2 > 0$. In this case, we have $\lambda_2 I_{N-1} \leq M_2^\intercal \text{Sym}(L) M_2 \leq \lambda_N I_N$.

III. Problem Formulation

Consider a collection of heterogeneous high-order nonlinear systems described by

$$\begin{aligned} \dot{x}_{j,i} &= x_{j+1,i} \\ \dot{x}_{n_i,i} &= \Delta_i(x_i, t) + u_i \\ y_i &= x_{1,i}, \quad i = 1, \ldots, N, \; j = 1, \ldots, n_i - 1 \end{aligned} \tag{1}$$

where $x_{j,i} \in \mathbb{R}$ is the jth state variable of agent i, $x_i \triangleq \text{col}(x_{1,i}, \ldots, x_{n_i,i}) \in \mathbb{R}^{n_i}$, $y_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are, respectively, the state, output, and input of agent i. Here, the term $\Delta_i(x_i, t)$ represents the unknown time-varying nonlinearity, which might result from modeling errors or external perturbations.

Suppose these agents play an N-player noncooperative game defined as follows. Agent i is endowed with a continuously differentiable cost function $J_i(y_i, y_{-i})$, where $y_i \in \mathbb{R}$ denotes the output strategy profile of agent i specified by (1) and $y_{-i} \in \mathbb{R}^{N-1}$ denote the output strategy profile of this multiagent system except for agent i. Each agent can change its output strategy profile according to (1) by specifying different actions (i.e., the control inputs). In this game, all agents seek to asymptotically minimize its own cost function J_i by reaching some proper steady-state output strategy. The equilibrium point of this game is defined as in [1] and [11].

Definition 1: Consider the game $\text{G} = \{\mathscr{N}, J_i, \mathbb{R}\}$. A strategy profile $y^* = \text{col}(y_1^*, \ldots, y_N^*)$ is said to be a Nash equilibrium of G if $J_i(y_i^*, y_{-i}^*) \leq J_i(y_i, y_{-i}^*)$ for any $i \in \mathscr{N}$ and $y_i \in \mathbb{R}$.

At a Nash equilibrium of the game G, all agents tend to keep at this state since no player can unilaterally decrease its cost by changing the steady-state output strategy on its own. Denote $\nabla_i J_i(y_i, y_{-i}) \triangleq \frac{\partial}{\partial y_i} J_i(y_i, y_{-i}) \in \mathbb{R}$ and $F(y) \triangleq \text{col}(\nabla_1 J_1(y_1, y_{-1}), \ldots, \nabla_N J_N(y_N, y_{-N})) \in \mathbb{R}^N$. Here, F is called the pseudogradient associated with $J_1, \ldots, J_N$.

The following assumption is often made in Nash equilibrium seeking literature [11], [12], [22].

Assumption 1: For any $i \in \mathscr{N}$, function $J_i(y_i, y_{-i})$ is continuously differentiable, and the associated pseudogradient F is $\underline{l}$-strongly monotone and $\bar{l}$-Lipschitz for some constants $\underline{l}, \bar{l} > 0$.

Under this assumption, this noncooperative game G admits a unique Nash equilibrium y^* characterized by the equation $F(y^*) = \mathbf{0}$ according to [27], Propositions 1.4.2 and 2.2.7]. To seek this Nash equilibrium y^*, it is equivalent for us to drive all agents to some steady-state with output profile y^*. We are interested in distributed designs where each agent can share information with a subset of the overall agents. For this purpose, a digraph $\mathscr{G} = (\mathscr{N}, \mathscr{E}, \mathscr{A})$ is used to describe the information flow among these agents. An edge (i, j) in digraph $\mathscr{G}$ means that agent j can the information of agent j. Here is a widely-used assumption in multiagent papers [9], [11], [28].

Assumption 2: Digraph $\mathscr{G}$ is weight-balanced and strongly connected.

We restrict in distributed controllers of the following form:

$$\begin{aligned} u_i &= k_i^{\text{s}}(\nabla J_i, x_i, z_j^{\text{s}}) \\ \dot{z}_i^{\text{s}} &= g_i^{\text{s}}(\nabla J_i, x_i, z_j^{\text{s}}), \quad j \in N_i \cup \{i\} \end{aligned} \tag{2}$$

with z_i^{s} the local compensator state and functions k_i^{s}, g_i^{s} to be specified later. It can be found that the information of ∇J_i and agent state x_i are private to each agent i in the gradient-play protocol and the agents can share information by communicating the local compensator states.

With these preparations, the distributed Nash equilibrium seeking problem in this article is formulated as follows.

Problem 1: For given multiagent system (1), digraph $\mathscr{G}$, and function J_i, determine a distributed protocol u_i of the form (2) for agent i such that

1) all trajectories of the closed-loop system are bounded over the time interval $[0, +\infty)$;
2) the outputs of agents satisfy $\lim_{t\to+\infty} ||y_i(t) - y_i^*|| = 0$ for any $i \in \mathscr{N}$ with $y^* = \text{col}(y_1^*, \ldots, y_N^*)$ being the Nash equilibrium of game G.

Remark 1: The formulated problem has been studied by many authors when the agent dynamics are restricted to single and/or multiple integrators. In contrast with existing works [9], [11], [18], [20], the considered agents in this article are allowed to be high-order and heterogeneous subject to unknown dynamics. These features make our problem much more challenging than the Nash equilibrium seeking works for integrators.

To ensure the solvability of this problem, we make an extra assumption on the unknown time-varying nonlinearity.

Assumption 3: For each $i \in \mathscr{N}$, there exists a known basis function vector $\boldsymbol{p}_i(x_i, t)$ such that $\Delta_i(x_i, t) \triangleq \theta_i^\top \boldsymbol{p}_i(x_i, t)$ for an uncertain parameter vector $\theta_i = \text{col}(\theta_{1,i}, \ldots, \theta_{n_{\theta_i},i}) \in \mathbb{R}^{n_{\theta_i}}$. Moreover, the basis function vector $\boldsymbol{p}_i(x_i, t)$ can be uniformly bounded by smooth functions of x_i.

Remark 2: This assumption is known as a linearly parameterized condition and has been intensively used in the literature [29], [30]. Equation (1) under Assumption 3 can represent a plenty of practical systems and is general enough to cover Van der Pol oscillators, Duffing equations and many mechanical systems [31]. When $n_{\theta_i} = 0$, the unknown nonlinearity vanishes and this class of agent dynamics include both single and multiple integrators as special cases.

As mentioned above, this problem has been partially investigated in the literature. However, due to the difficulty brought by the couplings among the high-order structure, information constraints, and the global equilibrium regulation requirement, these works are mostly derived for either integrator-type multiagent systems or undirected graphs case by case. Inspired by the embedded control scheme developed in [26] and [32] to solve distributed optimization problems, we borrow this decoupling idea to reduce these design complexities and extend it to solve the formulated Nash equilibrium seeking problem for high-order multiagent systems (1) in the following section.

IV. Main Result

In this section, we will detail the main design for solving Problem 1 by novel adaptive controllers along with a real-time gradient extension and parameter convergence analysis.

A. Embedded Design and Problem Conversion

Motivated by the designs in [26] and [32], we first consider some virtual multiagent system

$$\dot{z}_i = \mu_i, \quad i \in \mathscr{N}. \tag{3}$$

Suppose each virtual agent i plays the same noncooperative game as in Problem 1 with input μ_i and output strategy profile z_i. Here, each virtual player can be understood as an abstraction of the original agent (1) as discussed in [33] and [34]. By definition, the virtual game has the same Nash equilibrium y^* with our problem (1). If the virtual noncooperative game is solved, we only have to drive agent (1) to track z_i to reach the expected Nash equilibrium point. Since we work with digraphs, the Laplacian matrix L can be asymmetric. Existing algorithms derived for undirected graphs might fail to reproduce the expected Nash equilibrium [9], [11], [12], [13]. For this purpose, we provide a modified gradient-play rule for the virtual agent i

$$\begin{aligned} \dot{z}_i &= -\alpha \sum_{j=1}^{N} a_{ij} \left(z_i - z_i^j \right) - \nabla_i J_i \left(\mathbf{z}^i \right) \\ \dot{z}_k^i &= -\alpha \sum_{k=1}^{N} a_{ij} \left(z_k^i - z_k^j \right), \quad k \in \mathscr{N} \backslash \{i\} \end{aligned} \tag{4}$$

where $\mathbf{z}^i = \text{col}(z_1^i, \ldots, z_N^i)$ represents agent i's estimate of all virtual agents' strategies with $z_i^i = z_i$ and the constant $\alpha > 0$ is a proportional gain to be specified later. The function $\nabla_i J_i(\mathbf{z}^i) = \frac{\partial J_i}{\partial z_i^i}(\mathbf{z}^i)$ is the partial gradient of J_i evaluated at the local estimate $\mathbf{z}^i$. When $\alpha = 1$, system (5) reduces to the consensus-based gradient-play dynamics in [11]. Here, we add an adjustable parameter α to increase the gain of the consensus term to ensure its efficiency for weight-balanced digraphs.

For convenience, we define a vector-valued function $\boldsymbol{F}(\mathbf{z}) = \text{col}(\nabla_1 J_1(\mathbf{z}^1), \ldots, \nabla_N J_N(\mathbf{z}^N)) \in \mathbb{R}^N$. It is the extended pseudogradient of the cost functions $J_1, \ldots, J_N$. The following assumption has been widely used in the literature [11], [12].

Assumption 4: Function $\boldsymbol{F}$ is l_F-Lipschitz with $l_F > 0$.

Putting (4) into a compact form gives

$$\dot{\mathbf{z}} = -\alpha \mathbf{L}\mathbf{z} - R\boldsymbol{F}(\mathbf{z}) \tag{5}$$

with $\mathbf{z} = \text{col}(\mathbf{z}^1, \ldots, \mathbf{z}^N)$, $R = \text{diag}(R_1, \ldots, R_N)$, $R_i = \text{col}(\mathbf{0}_{i-1}, 1, \mathbf{0}_{N-i})$, and $\mathbf{L} = L \otimes I_N$. Denote $l = \max\{\bar{l}, l_F\}$. With α being large enough, the effectiveness of algorithm (5) has been established in [35]. Here, we provide a sketch of proof for completeness.

Lemma 1: Suppose Assumptions 1–4 hold. Let

$$\alpha > \frac{1}{\lambda_2}\left(\frac{l^2}{\underline{l}} + l\right) \tag{6}$$

Then, for any $i \in \mathcal{N}$, along the trajectory of system (5), $\boldsymbol{z}^i(t)$ exponentially converges to y^* as t goes to $+\infty$.

Proof: First, we can show that at the equilibrium of (5), z_i indeed reaches the Nash equilibrium of game G. In fact, letting the righthand side of (4) be zero and premultiplying both sides by $\mathbf{1}_N^\top \otimes I_N$, we have

$$\mathbf{0} = \alpha\left(\mathbf{1}_N^\top \otimes I_N\right)(L \otimes I_N)\mathbf{z}^* + \left(\mathbf{1}_N^\top \otimes I_N\right) R\boldsymbol{F}\left(\mathbf{z}^*\right).$$

By the fact $\mathbf{1}_N^\top L = 0$ under Assumption 2 and the notations of R and $\boldsymbol{F}$, we further have $\boldsymbol{F}(\mathbf{z}^*) = \mathbf{0}$ and $\mathbf{L}\mathbf{z}^* = \mathbf{0}$. Hence, there must be some $\theta \in \mathbb{R}^N$ such that $\mathbf{z}^* = \mathbf{1} \otimes \theta$ and $\boldsymbol{F}(\mathbf{1} \otimes \theta) = \mathbf{0}$, or equivalently, $F(\theta) = \mathbf{0}$. That is, θ is the unique Nash equilibrium y^* of G and $\mathbf{z}^* = \mathbf{1} \otimes y^*$.

Next, we show the exponential stability of (5) at its equilibrium $\mathbf{z}^* = \mathbf{1} \otimes y^*$. We denote $\tilde{\mathbf{z}} = \mathbf{z} - \mathbf{z}^*$ and perform the coordinate transformation $\overline{\mathbf{z}}_1 = (M_1^\top \otimes I_N)\tilde{\mathbf{z}}$ and $\overline{\mathbf{z}}_2 = (M_2^\top \otimes I_N)\tilde{\mathbf{z}}$ with M_1 and M_2 defined in Section II. It follows then

$$\begin{aligned}\dot{\overline{\mathbf{z}}}_1 &= -\left(M_1^\top \otimes I_N\right) R\Delta \\ \dot{\overline{\mathbf{z}}}_2 &= -\alpha[(M_2^\top L M_2) \otimes I_N]\overline{\mathbf{z}}_2 - \left(M_2^\top \otimes I_N\right) R\Delta\end{aligned}$$

where $\Delta \triangleq \boldsymbol{F}(\mathbf{z}) - \boldsymbol{F}(\mathbf{z}^*)$.

Let $V_0(\overline{\mathbf{z}}_1, \overline{\mathbf{z}}_2) = \frac{1}{2}(||\overline{\mathbf{z}}_1||^2 + ||\overline{\mathbf{z}}_2||^2)$. Its time derivative along the trajectory of system (5) satisfies that

$$\begin{aligned}\dot{V}_0 &= -\overline{\mathbf{z}}_1^\top \left(M_1^\top \otimes I_N\right) R\Delta - \overline{\mathbf{z}}_2^\top \left(M_2^\top \otimes I_N\right) R\Delta \\ &\quad - \alpha \overline{\mathbf{z}}_2^\top \{[M_2^\top L M_2] \otimes I_N\}\overline{\mathbf{z}}_2 \\ &\leq -\alpha\lambda_2||\overline{\mathbf{z}}_2||^2 - \tilde{\mathbf{z}}^\top R\Delta.\end{aligned} \tag{7}$$

Jointly using the facts $\boldsymbol{F}(\mathbf{1}_N \otimes y) = F(y)$ for any $y \in \mathbb{R}^N$, $(\mathbf{1}^\top \otimes I_N)R = I_N$, and $\tilde{\mathbf{z}}_1^\top R = \frac{\overline{\mathbf{z}}_1^\top}{\sqrt{N}}$, we can bound the cross term by Young's inequality and have that

$$-\tilde{\mathbf{z}}^\top R\Delta \leq \frac{2\,l}{\sqrt{N}}||\overline{\mathbf{z}}_1||||\overline{\mathbf{z}}_2|| + l||\overline{\mathbf{z}}_2||^2 - \frac{\underline{l}}{N}||\overline{\mathbf{z}}_1||^2. \tag{8}$$

Bringing inequalities (7) and (8) together gives

$$\begin{aligned}\dot{V}_0 &\leq -\frac{\underline{l}}{N}||\overline{\mathbf{z}}_1||^2 - (\alpha\lambda_2 - l)||\overline{\mathbf{z}}_2||^2 + \frac{2\,l}{\sqrt{N}}||\overline{\mathbf{z}}_1||||\overline{\mathbf{z}}_2|| \\ &= -\begin{bmatrix}||\overline{\mathbf{z}}_1|| & ||\overline{\mathbf{z}}_2||\end{bmatrix} A_\alpha \begin{bmatrix}||\overline{\mathbf{z}}_1|| \\ ||\overline{\mathbf{z}}_2||\end{bmatrix}\end{aligned} \tag{9}$$

with $A_\alpha = \begin{bmatrix}\frac{\underline{l}}{N} & -\frac{l}{\sqrt{N}} \\ -\frac{l}{\sqrt{N}} & \alpha\lambda_2 - l\end{bmatrix}$. Under the choice (6), matrix A_α is verified to be positive definite. It follows then

$$\dot{V}_0 \leq -\nu V_0$$

with ν the minimal eigenvalue of A_α. Using [31], Th. 4.10], one can conclude the exponential convergence of $\mathbf{z}(t)$ to $\mathbf{z}^*$, which implies that $\mathbf{z}^i(t)$ is bounded over $[0, +\infty)$ and converges to z^* as $t \to +\infty$. ■

Remark 3: The criterion (6) to choose α clearly presents a natural tradeoff between the control efforts and graph algebraic connectivity. This observation is consistent with the results in [11] when α is fixed as one. By choosing a large enough α, this lemma provides an alternative way to remove the restrictive graph coupling condition other than singular perturbation analysis in [11].

Remark 4: In our design, we use some information of the communication graph and agents' cost functions as in [11], [12], and [22] to ensure the exponential convergence of virtual agent's state toward the Nash equilibrium y^* under weight-balanced digraphs. In practice, we may compute these values beforehand by existing algorithms (e.g., [36]) or select α from numerical simulations by repeatedly increasing it.

Next, we are left to solve the output tracking problem for agent (1) with reference $z_i(t)$ generated by (4). Compared with classical tracking control settings [29], [31], we will not use the derivative of $z_i(t)$ and leave it as a chosen input.

Due to the presence of uncertain parameter θ_i, direct cancellation cannot be used to handle the nonlinearities in (1). To tackle this issue, we adopt a certainty-equivalence design and propose the following tracking controller for each agent:

$$\begin{aligned}u_i &= -\hat{\theta}_i^\top \boldsymbol{p}_i(x_i, t) + \frac{1}{\epsilon^{n_i}}\left[k_{1i}(x_{1,i} - z_i) + \sum_{j=2}^{n_i} \epsilon^{j-1} k_{ji} x_{j,i}\right] \\ \dot{\hat{\theta}}_i &= \phi_i(x_i, \hat{\theta}_i, z_i, t)\end{aligned} \tag{10}$$

where $\hat{\theta}_i$ is the estimation of θ_i with constants $\epsilon, k_{1i}, \ldots, k_{n_i i} > 0$ and smooth function $\phi_i(\cdot)$ to be specified later. Then, the final controller to solve Problem 1 is composed of (4) and (10), which is exactly of the form (2). We sketch the structure of the whole closed-loop system with (4) and (10) in Fig. 1.

Under the above control law, we have

$$\begin{aligned}\dot{x}_{1,i} &= x_{2,i} \\ &\vdots \\ \dot{x}_{n_i,i} &= (\theta_i^\top - \hat{\theta}_i^\top)\boldsymbol{p}_i(x_i, t) + \frac{1}{\epsilon^{n_i}}\Big[k_{1i}(x_{1,i} - z_i)\end{aligned}$$

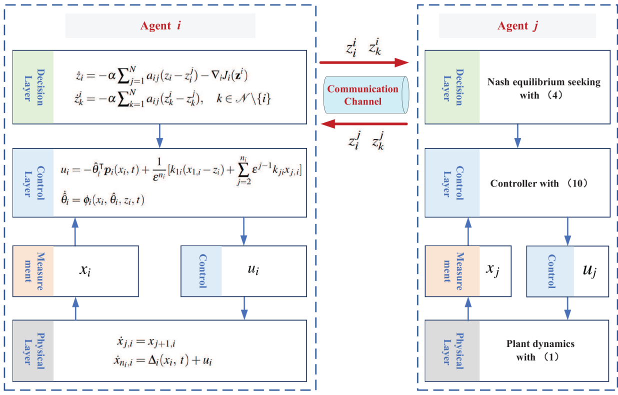

Fig. 1. Block diagram of closed-loop system with (4) and (10).

$$
\left. + \sum_{j=2}^{n_i} \epsilon^{j-1} k_{ji} x_{j,\,i} \right]
$$

$$
\begin{aligned}
\dot{\hat{\theta}}_i &= \phi_i(x_i,\, \hat{\theta}_i,\, z_i,\, t)\\
\dot{z}_i &= -\alpha \sum_{j=1}^{N} a_{ij}(z_i - z_i^j) - \nabla_i J_i(\mathbf{z}^i)\\
\dot{z}_k^i &= -\alpha \sum_{k=1}^{N} a_{ij}(z_k^i - z_k^j), \quad k \in \mathcal{N}\backslash\{i\}.
\end{aligned}
\tag{11}
$$

Letting $\hat{x}_i = \mathrm{col}(x_{1,i} - z_i,\, \epsilon x_{2,i},\, \ldots,\, \epsilon^{n_i - 1} x_{n_i,i})$, agent (1) under the controller (4) and (10) can be rewritten as follows:

$$
\begin{aligned}
\epsilon\dot{\hat{x}}_i &= A_i \hat{x}_i - \epsilon b_{1i} \dot{z}_i + \epsilon^{n_i} b_{2i}(\theta_i^\mathsf{T} - \hat{\theta}_i^\mathsf{T})\boldsymbol{p}_i(x_i,\, t)\\
\dot{\hat{\theta}}_i &= \phi_i(x_i,\, \hat{\theta}_i,\, z_i,\, t)\\
\dot{z}_i &= -\alpha \sum_{j=1}^{N} a_{ij}(z_i - z_i^j) - \nabla_i J_i(\mathbf{z}^i)\\
\dot{z}_k^i &= -\alpha \sum_{k=1}^{N} a_{ij}(z_k^i - z_k^j), \quad k \in \mathcal{N}\backslash\{i\}
\end{aligned}
\tag{12}
$$

where $A_i = \left[\begin{array}{c|c} \mathbf{0} & I_{n_{x_i}-1} \\ \hline k_{1i} & [k_{2i} \ \ldots \ k_{n_i\,i}] \end{array}\right]$, $b_{1i} = \mathrm{col}(1,\, \mathbf{0})$, and $b_{2i} = \mathrm{col}(\mathbf{0},\, 1)$.

We first choose constants $k_{1i},\, \ldots,\, k_{n_i\,i}$ such that the polynomial $s^{n_i} - k_{n_i\,i} s^{n_i - 1} - \cdots - k_{2i} s - k_{1i}$ is Hurwitz for any $1 \le i \le N$. Then, the Lyapunov equation $A_i^\mathsf{T} P_i + P_i A_i = -2 I_{n_i}$ has a unique positive definite solution P_i with compatible dimensions for any $i \in \mathcal{N}$. Based on the above observations, we only need to determine some proper function $\phi_i(\cdot)$ such that all trajectories of (12) is bounded over $[0,\, +\infty)$ and satisfying $\hat{x}_i(t) \to 0$ as t goes to infinity.

B. Solvability Analysis

Denote $\hat{x} = \mathrm{col}(\hat{x}_1,\, \ldots,\, \hat{x}_N)$, $\theta = \mathrm{col}(\theta_1,\, \ldots,\, \theta_N)$, $\hat{\theta} = \mathrm{col}(\hat{\theta}_1,\, \ldots,\, \hat{\theta}_N)$, $\overline{\theta} = \theta - \hat{\theta}$, and $z = \mathrm{col}(z_1,\, \ldots,\, z_N)$ for short. The whole composite multiagent system can be put into a compact form as follows:

$$
\begin{aligned}
\dot{\hat{x}} &= \frac{1}{\epsilon} A\hat{x} - B_1 \dot{z} + E B_2 \boldsymbol{p}^\mathsf{T}(x,\, t)\overline{\theta}\\
\dot{\hat{\theta}} &= \phi(x,\, \hat{\theta},\, z,\, t)\\
\dot{\mathbf{z}} &= -\alpha \mathbf{L}\mathbf{z} - R\boldsymbol{F}(\mathbf{z})
\end{aligned}
\tag{13}
$$

where

$$
\begin{aligned}
A &\triangleq \mathrm{blockdiag}(A_1,\, \ldots,\, A_N)\\
B_1 &\triangleq \mathrm{blockdiag}(b_{11},\, \ldots,\, b_{1\,N})\\
B_2 &\triangleq \mathrm{blockdiag}(b_{21},\, \ldots,\, b_{2\,N})\\
E &\triangleq \mathrm{blockdiag}\left(\epsilon^{n_1 - 1} I_{n_1},\, \ldots,\, \epsilon^{n_N - 1} I_{n_N}\right)\\
\phi(x,\, \hat{\theta},\, z,\, t) &\triangleq \mathrm{col}\Big(\phi_1(x_1,\, \hat{\theta}_1,\, z_1,\, t),\, \ldots,\\
&\qquad \phi_N\left(x_N,\, \hat{\theta}_N,\, z_N,\, t\right)\Big)\\
\boldsymbol{p}(x,\, t) &\triangleq \mathrm{blockdiag}(\boldsymbol{p}_1(x_1,\, t),\, \ldots,\, \boldsymbol{p}_N(x_N,\, t)).
\end{aligned}
$$

Here is the first main theorem of this article.

Theorem 1: Suppose Assumptions 1–4 hold. Choose constants $k_{1i},\, \ldots,\, k_{n_i\,i}$ such that the polynomial $s^{n_i} - k_{n_i\,i} s^{n_i - 1} - \cdots - k_{2i} s - k_{1i}$ is Hurwitz for each $i \in \mathcal{N}$ and let $\alpha > \frac{1}{\lambda_2}(\frac{l^2}{\underline{l}} + l)$. Then, Problem 1 for multiagent system (1) is solved by a distributed controller composed of (4) and (10) with $\phi_i(x_i,\, \hat{\theta}_i,\, z_i,\, t) = \boldsymbol{p}_i(x_i,\, t) b_{2i}^\mathsf{T} P_i \hat{x}_i$ for any $\epsilon > 0$.

Proof: Under the theorem conditions, we can recall Lemma 1 and conclude the exponential convergence of $z_i(t)$ toward y_i^*. Then, it is sufficient for us to prove $\hat{x}_i(t) \to 0$ as $t \to +\infty$ to ensure that $\lim_{t \to +\infty}[y_i(t) - z_i(t)] = 0$. To this end, we present a Lyapunov analysis for system (13).

Let us consider the first two subsystems of (13). Let $\hat{V}_i = \hat{W}_i + \epsilon^{n_i - 1} \overline{\theta}_i^\mathsf{T} \overline{\theta}_i$ with $\hat{W}_i = \hat{x}_i^\mathsf{T} P_i \hat{x}_i$. Its time derivative along the trajectory of the composite system (13) satisfies

$$
\begin{aligned}
\dot{\hat{V}}_i &= 2\hat{x}_i^\mathsf{T} P_i \left[\frac{1}{\epsilon} A_i \hat{x}_i - b_{1i} \dot{z}_i + \epsilon^{n_i - 1} b_{2i} \overline{\theta}_i^\mathsf{T} \boldsymbol{p}_i(x_i,\, t)\right]\\
&\quad - 2\epsilon^{n_i - 1} \overline{\theta}_i^\mathsf{T} \phi_i\left(x_i,\, \hat{\theta}_i,\, z_i,\, t\right)\\
&= -\frac{2}{\epsilon} \hat{x}_i^\mathsf{T} \hat{x}_i - 2\hat{x}_i^\mathsf{T} P_i b_{1i} \dot{z}_i.
\end{aligned}
$$

By Young's inequality, it holds that

$$
\begin{aligned}
\dot{\hat{V}}_i &\le -\frac{2}{\epsilon} \hat{x}_i^\mathsf{T} \hat{x}_i + \frac{1}{\epsilon} ||\hat{x}_i||^2 + \epsilon ||P_i b_{1i}||^2 ||\dot{z}_i||^2\\
&\le -\frac{1}{\epsilon} ||\hat{x}_i||^2 + c_1 ||\dot{z}_i||^2
\end{aligned}
$$

where $c_1 = \max_{i \in \mathcal{N}} \epsilon ||P_i b_{1i}||^2$.

At the same time, we can determine a quadratic Lyapunov function $V_0(\overline{\mathbf{z}})$ according to Lemma 1 or its proof such that

$$
\dot{V}_0(t) \le -\nu V_0
$$

for some constant $\nu > 0$ with $\overline{\mathbf{z}} = \mathbf{z} - \mathbf{1} \otimes y^*$. Under Assumption 4, the righthand side of system (5) is globally Lipschitz.

Thus, there exists a constant $c_2 > 0$ such that $||\dot{z}_i||^2 \leq ||\dot{\mathbf{z}}||^2 \leq c_2 V_0(\overline{\mathbf{z}})$ along the trajectory of system (5).

Next, we choose a Lyapunov function for the whole composite system (13) perhaps after some coordinate transformation of $\mathbf{z}$ to $\overline{\mathbf{z}}$ as $\hat{V} = \sum_{i=1}^N \hat{V}_i + c_3 V_0$ with $c_3 > 0$ to be specified later. It is positive definite and radially unbounded. Combining the above inequalities, one has

$$\begin{aligned}\dot{\hat{V}} \leq & -\sum_{i=1}^N \frac{1}{\epsilon}||\hat{x}_i||^2 + c_1 \sum_{i=1}^N ||\dot{z}_i||^2 - c_3 \nu V_0 \\ \leq & -\frac{1}{\epsilon}||\hat{x}||^2 + c_1||\dot{z}||^2 - c_3\nu V_0 \\ \leq & -\frac{1}{\epsilon}||\hat{x}||^2 - (c_3\nu - c_1 c_2)V_0.\end{aligned}$$

Letting $c_3 > \frac{c_1 c_2 + 1}{\nu}$ gives

$$\dot{\hat{V}} \leq -\frac{1}{\epsilon}||\hat{x}||^2 - V_0 \triangleq \hat{W}(\hat{x},\ \overline{\mathbf{z}}). \tag{14}$$

Recalling [29], Th. 2.1], we have that all trajectories of the closed-loop system (13) are bounded over the time interval $[0, +\infty)$ and satisfy that $\lim_{t\to\infty} ||\hat{x}_i(t)|| = 0$. As immediate results, one can conclude the boundedness of $x_{j,i}(t)$ and $\theta(t)$. Moreover, we can obtain that $\lim_{t\to\infty} \hat{x}_{1,i}(t) = 0$, that is, $\lim_{t\to\infty}[y_i(t) - z_i(t)] = 0$. Using the triangle inequality, we have $|y_i(t) - y_i^*| \leq |y_i(t) - z_i(t)| + |z_i(t) - y_i^*| \to 0$ as $t \to +\infty$. ∎

Remark 5: Note that the considered agent (1) is high-order and subject to unknown dynamics, which includes both single and multiple integrators as its special cases. Thus, the theorem can be taken as an adaptive extension to existing results when the agent dynamics are exactly known [11], [18]. Moreover, many typical actuating disturbances can be represented by the form (1) including the case when the disturbance is generated by a known autonomous linear dynamics as in [18] and [37]. Thus, we provide an alternative way to reject external disturbances other than the observer-based approach used in [18] and the internal model-based design in [21] and [23].

Remark 6: In the developed controller (10), we may choose $\phi_i(x_i,\ \hat{\theta}_i,\ z_i,\ t) = \Lambda_i \boldsymbol{p}_i(x_i,\ t) b_{2i}^\intercal P_i \hat{x}_i$ with Λ_i a chosen positive definite matrix. This matrix Λ_i is called the adaption gain in the literature [38]. It can be used to achieve a fast adaption and then improve the transient performance of the controller to solve our Nash equilibrium seeking problem.

C. Real-Time Gradient Extension

In the preceding section, we implicitly assume the partial gradient function $\nabla_i J_i$ can be evaluated at any given estimate $\mathbf{z}^i$. This is the case when the analytic form of the local cost function is known by agent i. However, in many circumstances, we may not have this knowledge and such partial gradient information can only be accessed or approximated when the real-time output strategy y_i is taken [2]. In this case, the generator (4) fails to be implemented.

Let us replace $\nabla_i J_i(\mathbf{z}^i)$ by $\nabla_i J_i(y_i,\ \mathbf{z}_{-i}^i)$ and obtain

$$\begin{aligned}\dot{z}_i &= -\alpha \sum_{j=1}^N a_{ij}\left(z_i - z_i^j\right) - \nabla_i J_i\left(y_i,\ \mathbf{z}_{-i}^i\right) \\ \dot{z}_k^i &= -\alpha \sum_{k=1}^N a_{ij}\left(z_k^i - z_k^j\right), \quad k \in \mathscr{N}\backslash\{i\}.\end{aligned} \tag{15}$$

Although this dynamics is similar with (4), it cannot generate the expected Nash equilibrium by itself. In fact, denoting $\Delta_i^1 \triangleq \nabla_i J_i(\mathbf{z}^i) - \nabla_i J_i(y_i,\ \mathbf{z}_{-i}^i)$ gives

$$\begin{aligned}\dot{z}_i &= -\alpha \sum_{j=1}^N a_{ij}\left(z_i - z_i^j\right) - \nabla_i J_i(\mathbf{z}_i) - \Delta_i^1 \\ \dot{z}_k^i &= -\alpha \sum_{k=1}^N a_{ij}\left(z_k^i - z_k^j\right), \quad k \in \mathscr{N}\backslash\{i\}.\end{aligned}$$

Compared with the optimal signal generator (4), the error term Δ_i^1 always exists except the case when $x_{1,i} = z_i$. Thus, there must be a discrepancy between $\boldsymbol{z}_i$ and z^* when $\Delta_i^1 \neq 0$.

Putting (16) into a compact form, we have

$$\dot{\mathbf{z}} = -\alpha \mathbf{L}\mathbf{z} - R\boldsymbol{F}(\mathbf{z}) + R\Delta^1 \tag{16}$$

where $\Delta^1 = \mathrm{col}(\Delta_1^1,\ \ldots,\ \Delta_N^1)$. Note that system (16) is exponentially stable when $\Delta^1 \equiv \mathbf{0}$ by Lemma 1. At the same time, one can verify that Δ_i^1 is l-Lipschitz with respect to the tracking error $x_{1,i} - z_i$ (or $\tilde{x}_i$) for each $i \in \mathscr{N}$ by Assumption 4. These facts inspire us to develop fast-tracking controllers for each agent to compensate the estimate error $\mathbf{z}_i - z^*$ and complete the whole design by decreasing the parameter ϵ.

To this end, we use the same tracking controller for each agent as in the previous subsection. Jointly with the modified generator (15), the overall controller to solve our problem using only real-time gradients is presented as follows:

$$\begin{aligned}u_i &= -\hat{\theta}_i^\intercal \boldsymbol{p}_i(x_i,\ t) + \frac{1}{\epsilon^{n_i}}\left[k_{1i}(x_{1,i} - z_i) + \sum_{j=2}^{n_i} \epsilon^{j-1} k_{ji} x_{j,i}\right] \\ \dot{\hat{\theta}}_i &= \phi_i\left(x_i,\ \hat{\theta}_i,\ z_i,\ t\right) \\ \dot{z}_i &= -\alpha \sum_{j=1}^N a_{ij}\left(z_i - z_i^j\right) - \nabla_i J_i\left(y_i,\ \mathbf{z}_{-i}^i\right) \\ \dot{z}_k^i &= -\alpha \sum_{k=1}^N a_{ij}\left(z_k^i - z_k^j\right), \quad k \in \mathscr{N}\backslash\{i\}\end{aligned} \tag{17}$$

with $\epsilon > 0$ to be specified later. The structure of the whole closed-loop system under this controller is shown in Fig. 2.

Here is a theorem to ensure the effectiveness of this controller to solve our problem.

Theorem 2: Suppose Assumptions 1–4 hold. Choose α, k_{1i}, $\ldots$ $k_{n_i i}$, and ϕ_i as in Theorem 1 for each $i \in \mathscr{N}$. Then, there exists a constant $\epsilon^* > 0$ such that the Nash equilibrium seeking problem for multiagent system (1) is solved by distributed controllers of the form (17) for any $\epsilon \in (0,\ \epsilon^*)$.

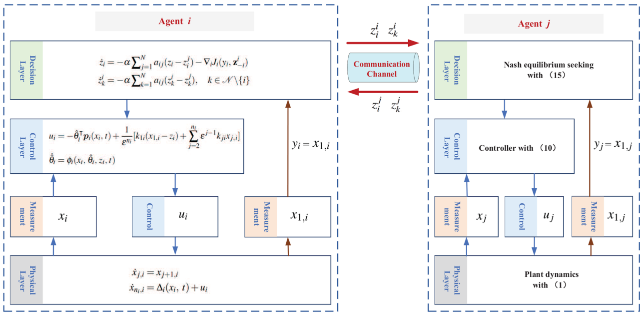

Fig. 2. Block diagram of closed-loop system with (15) and (10).

Proof: Under the new controller (17), the whole composite system is then

$$
\begin{aligned}
\dot{x}_{1,i} &= x_{2,i}\\
&\ \ \vdots\\
\dot{x}_{n_i,i} &= \left(\theta_i^\intercal - \hat{\theta}_i^\intercal\right)\boldsymbol{p}_i(x_i,\,t)\\
&\quad + \frac{1}{\epsilon^{n_i}}\left[k_{1i}(x_{1,i}-z_i) + \sum_{j=2}^{n_i}\epsilon^{j-1}k_{ji}x_{j,i}\right]\\
\dot{\hat{\theta}}_i &= \phi_i\left(x_i,\,\hat{\theta}_i,\,z_i,\,t\right)\\
\dot{z}_i &= -\alpha\sum_{j=1}^{N}a_{ij}(z_i - z_i^j) - \nabla_i J_i\left(y_i,\,\mathbf{z}_{-i}^i\right)\\
\dot{z}_k^i &= -\alpha\sum_{k=1}^{N}a_{ij}\left(z_k^i - z_k^j\right),\quad k\in\mathscr{N}\backslash\{i\}.
\end{aligned}
$$

By applying the same transformation of coordinates as in system (12), we can put the above composite system into a compact form as follows:

$$
\begin{aligned}
\dot{\hat{x}} &= \frac{1}{\epsilon}A\hat{x} - B_1\dot{z} + EB_2\boldsymbol{p}^\intercal(x,\,t)\overline{\theta}\\
\dot{\overline{\theta}} &= \phi\left(x,\,\hat{\theta},\,z,\,t\right)\\
\dot{\mathbf{z}} &= -\alpha\mathbf{L}\mathbf{z} - R\boldsymbol{F}(\mathbf{z}) - R\Delta^1.
\end{aligned}
\tag{18}
$$

Note that the third subsystem can be further rewritten in the $(\overline{\mathbf{z}}_1,\,\overline{\mathbf{z}}_2)$ coordinate as

$$
\begin{aligned}
\dot{\overline{\mathbf{z}}}_1 &= -(M_1^\intercal\otimes I_N)R\Delta - (M_1^\intercal\otimes I_N)R\Delta^1\\
\dot{\overline{\mathbf{z}}}_2 &= -\alpha[(M_2^\intercal LM_2)\otimes I_N]\overline{\mathbf{z}}_2\\
&\quad - (M_2^\intercal\otimes I_N)R\Delta - (M_2^\intercal\otimes I_N)R\Delta^1.
\end{aligned}
$$

Next, we use the same Lyapunov function $\hat{V} = \sum_{i=1}^N \hat{V}_i + c_3V_0$ for this new composite system (13) with $c_3 > 0$ defined as in the proof of Theorem 1. Following a similar procedure in deriving (14) in the proof of Theorem 1, we have

$$
\begin{aligned}
\dot{\hat{V}} \le &-\sum_{i=1}^{N}\frac{1}{\epsilon}||\hat{x}_i||^2 + c_1\sum_{i=1}^{N}||\dot{z}_i||^2\\
&- c_3\overline{\mathbf{z}}_1^\intercal\left(M_1^\intercal\otimes I_N\right)R\left(\Delta+\Delta^1\right)\\
&- c_3\overline{\mathbf{z}}_2^\intercal\{\alpha\left[(M_2^\intercal LM_2)\otimes I_N\right]\overline{\mathbf{z}}_2\\
&+ \left(M_2^\intercal\otimes I_N\right)R\left(\Delta+\Delta^1\right)\}\\
\le &-\sum_{i=1}^{N}\frac{1}{\epsilon}||\hat{x}_i||^2 + c_1\sum_{i=1}^{N}||\dot{z}_i||^2 - c_3\nu V_0 - c_3\tilde{\mathbf{z}}^\intercal R\Delta^1\\
\le &-\frac{1}{\epsilon}||\hat{x}||^2 - V_0 - c_3||\overline{\mathbf{z}}||||\Delta^1||
\end{aligned}
$$

where we use $||\tilde{\mathbf{z}}|| = ||\overline{\mathbf{z}}||$. By Young's inequality and the l-Lipschitzness of Δ^1 with respect to $\hat{x}$, it follows that

$$
\begin{aligned}
\dot{\hat{V}} &\le -\frac{1}{\epsilon}||\hat{x}||^2 - V_0 + \frac{1}{4}||\overline{\mathbf{z}}||^2 + 4c_3^2l^2||\hat{x}||^2\\
&\le -\left(\frac{1}{\epsilon} - 4c_3^2l^2\right)||\hat{x}||^2 - \frac{1}{2}V_0.
\end{aligned}
$$

Setting $\epsilon^* = \frac{1}{4c_3^2l^2+1}$ and $\epsilon\in(0,\,\epsilon^*)$, we can obtain the following inequality:

$$
\dot{\hat{V}} \le -||\hat{x}||^2 - \frac{1}{2}V_0
$$

At this moment, we recall [29], Th. 2.1] again and conclude that all trajectories of the closed-loop system (18) are bounded over the time interval $[0,\,+\infty)$ and satisfy that $\lim_{t\to\infty}||\hat{x}_i(t)|| = 0$ and $\lim_{t\to\infty}V_0(t) = 0$. Then, we confirm the boundedness of $x_{j,i}(t)$, $\theta(t)$ and conclude $\lim_{t\to\infty}[y_i(t) - z_i(t)] = 0$, $\lim_{t\to\infty}[z_i(t) - y^*] = 0$. By the triangle inequality again, it follows that $|y_i(t) - y_i^*| \le |y_i(t) - z_i(t)| + |z_i(t) - y_i^*| \to 0$ as $t\to+\infty$. ■

Remark 7: Theorem 2 shows that when a real-time gradient is applied, the algorithm can still have convergence properties. In that case, there exists a strongly coupling between the physical dynamics and information measurement (thus the gradient). It shows that even though the proposed embedded methodology takes a decoupling pipeline for algorithm design, it can also handle the cases when there are close bidirectional cyber-physical feedback connections.

Remark 8: The specified parameter ϵ is in a high-gain fashion to compensate for the interconnection between the virtual agent dynamic (15) and the tracking controller (10). The given choice of ϵ^* heavily relies on some bounds of matrix norms. Thus, determining the largest parameter ϵ^* in controller (17) might be nontrivial. In practice, one may choose an applicable parameter ϵ by numerical simulations to avoid this tedious job.

D. Parameter Convergence

From the proofs of Theorems 1 and 2, one can merely conclude that $\hat{\theta}_i(t)$ converges to some constant as t tends to $+\infty$. However, this constant may not be the associated true value θ_i. Since parameter convergence has been shown to be essential in achieving robustness of the adaptive controllers [38], [39], we assert conditions under which the estimator $\hat{\theta}_i(t)$ will converge to its true value θ_i as t tends to $+\infty$.

To this end, we further assume the basis function $\boldsymbol{p}_i(x_i,\,t)$ satisfies the following condition.

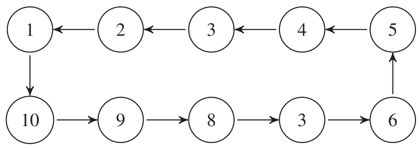

Fig. 3. Communication digraph $\mathcal{G}$ in Example I.

Assumption 5: For any $i = 1, \ldots, N$, along the trajectory of the closed-loop system composed of (1) and (10), there exist positive constants m, t_0, T_0 such that the function $\boldsymbol{p}_i(x_i(t), t)$ is uniformly bounded and the following inequality is satisfied:

$$\frac{1}{T_0}\int_t^{t+T_0} \boldsymbol{p}_i(x_i(\tau), \tau)\boldsymbol{p}^{\mathsf{T}}(x_i(\tau), \tau)\mathrm{d}\tau \geq mI_{n_{\theta_i}} \quad \forall t \geq t_0.$$

Note that $x_i(t)$ is ultimately bounded by Theorem 1, the boundedness of $\boldsymbol{p}_i(x_i(t), t)$ is not too strict. The above inequality is known as a version of the well-known persistence of excitation (PE) condition and has been widely used in adaptive control literature [29], [30], [40].

Theorem 3: Suppose Assumptions 1–5 hold. Then, along the trajectory of system (1) under the controllers proposed in Theorems 1 and 2, $\lim_{t\to+\infty}\hat{\theta}_i(t) = \theta_i$ holds for each $i \in \mathcal{N}$.

Proof: To show this theorem, we first claim that $\lim_{t\to+\infty}\overline{\theta}_i^{\mathsf{T}}(t)\boldsymbol{p}_i(x_i(t), t) = 0$. By the proof of Theorems 1 and 2, we have $\hat{x}_i(\infty) = \int_0^{+\infty}\dot{\hat{x}}_i(\tau)\mathrm{d}\tau = 0$. From the uniform boundedness of associated variables and Assumption 5, it follows that $\ddot{\hat{x}}_i(t)$ is also bounded. Using [31], Lemma 8.2] to $\dot{\hat{x}}_i(t)$ implies that $\lim_{t\to+\infty}\dot{\hat{x}}_i(t) = 0$, which confirms this claim.

Next, since $\dot{\overline{\theta}}_i = \dot{\hat{\theta}}_i = \boldsymbol{p}_i(x_i, t)b_{2i}^{\mathsf{T}}P_i\hat{x}_i$, it follows that $\lim_{t\to+\infty}\dot{\overline{\theta}}_i(t) = 0$. According to [41], Lemma 1], the two facts $\lim_{t\to+\infty}\dot{\overline{\theta}}_i(t) = 0$ and $\lim_{t\to+\infty}\overline{\theta}_i^{\mathsf{T}}(t)\boldsymbol{p}_i(x_i(t), t) = 0$ provide us that $\lim_{t\to+\infty}\overline{\theta}_i(t) = 0$ under Assumption 5. ■

Remark 9: Note that the unknown dynamics is supposed to be linearly parameterized in this article. This structure allows us to further improve Theorem 3 and apply it to any number of components in $\boldsymbol{p}_i(x_i, t)$ satisfying such a PE condition. In this way, we can address the parameter convergence problem in a more precise way. Specially, when the basis function is time-invariant, the jth component $\boldsymbol{p}_{j,i}(x_i)$ of $\boldsymbol{p}_i(x_i)$ is persistently excited if $\lim_{x_i\to\mathrm{col}(y_i^*, 0, \ldots, 0)}\boldsymbol{p}_{j,i}(x_i) \neq 0$, which ensures the convergence of $\hat{\theta}_{j,i}(t)$ to $\theta_{j,i}$ as t goes to infinity.

V. Simulation

In this section, we present numerical examples to illustrate the effectiveness of our preceding design.

Example 1: Consider a group of $N = 10$ firms and suppose they produce a homogeneous perishable commodity [42]. The inventory system at firm i can be modeled as

$$\dot{I}_i = -\gamma_i I_i + P_i - D_i, \quad i \in \mathcal{N} \tag{19}$$

where I_i is the inventory level, γ_i is the deterioration rate, P_i is the production rate, and D_i is demand rate at firm i. Suppose these firms can share information through a cycle digraph with unity weights depicted in Fig. 3.

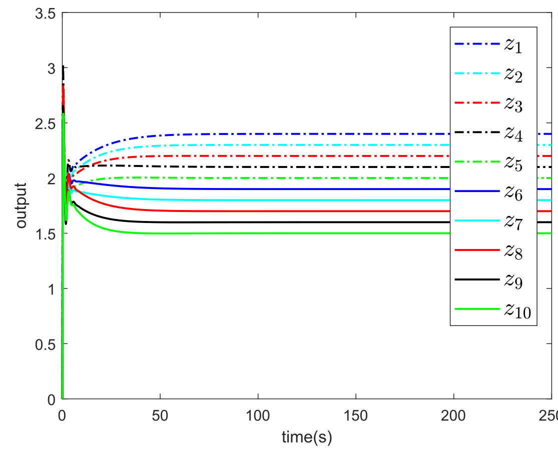

Fig. 4. Profiles of $z_i(t)$ under the algorithm (4) in *Example 1*.

To meet a safety requirement imposed by some authority in this market (e.g., the government), these firms are expected to maintain their total inventory at a certain level $I_r > 0$ by changing its production rate P_i. The total cost function of firm i is given as $J_i(I_i, I_{-i}) = C_i(I_i) - I_i * \sigma(I_1, \ldots, I_N)$, where $C_i(s) = \alpha_i s$ is the storage cost and $\sigma(I_1, \ldots, I_N) = \delta_0(I_r - \sum_{i=1}^N I_i)$ is the subsidies per unit provided by this market authority with known constants $\alpha_i, \delta_0 > 0$. To make it more interesting, we suppose that the deterioration rate γ_i and the demand rate D_i at agent i are both constant but unknown. Letting $\theta_i = \mathrm{col}(-\gamma_i, D_i)$ and $\boldsymbol{p}(I_i, t) = \mathrm{col}(-I_i, 1)$, we can find that these firms play a noncooperative game with cost function J_i and unknown dynamics of the form (19) with input P_i and output I_i. Moreover, Assumptions 1–4 can be practically verified. Then, according to Theorem 1, we can determine a distributed controller composed of (4) and (10) with $n_i = 1$ to solve the formulated problem for agent (19).

For simulations, we assume $N = 10$ and let $\alpha_i = i/10$, $I_r = 22$, $\delta_0 = 1$, $\theta_i = \frac{i}{2}$, and $D_i = 10 - i$. The Nash equilibrium is determined as $y^* = \mathrm{col}(y_1^*, \ldots, y_{10})$ with $y_i^* = 2.5 - \alpha_i$ for $i = 1, \ldots, N$. Choose $\alpha = 4$, $k_1 = -4$ for the controller (10) and the initial inventory levels randomly from $[0, 5]$. The generated reference for each agent is depicted in Fig. 4. To verify the efficiency of our controller in handing unknown dynamics, we shut down the adaptive part between $t = 100$s and $t = 150$s. The profiles of agent outputs and control efforts are shown in Figs. 5 and 6, where the expected Nash equilibrium y^* is quickly reached before $t = 100$s and the control efforts are maintained to be bounded. Moreover, we can find that the steady state of each agent deviates from the expected Nash equilibrium y^* after $t = 100$s and soon recovers after $t = 150$s.

Example 2: Consider the source-seeking problem via mobile sensor networks. Suppose we have a group of five force-actuated mobile robots in the plane modeled as follows:

$$\ddot{y}_i = u_i + d_i \tag{20}$$

where $y_i \in \mathbb{R}^2$, $\dot{y}_i \in \mathbb{R}^2$, and $u_i \in \mathbb{R}^2$ are, respectively, the position, velocity, and control input of agent i. Here, $d_i \in \mathbb{R}^2$ is a local actuating disturbance of agent i. The communication topology is represented by an undirected graph with unity edge

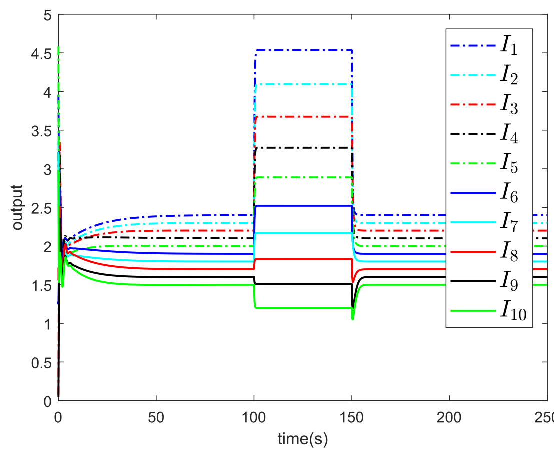

Fig. 5. Profiles of $I_i(t)$ under the controller (4) and (10) in *Example 1*.

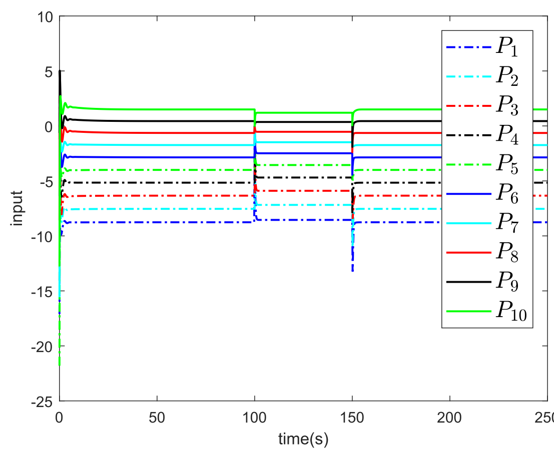

Fig. 6. Profiles of $P_i(t)$ under the controller (4) and (10) in *Example 1*.

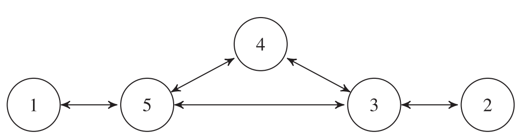

Fig. 7. Communication digraph $\mathcal{G}$ in *Example 2*.

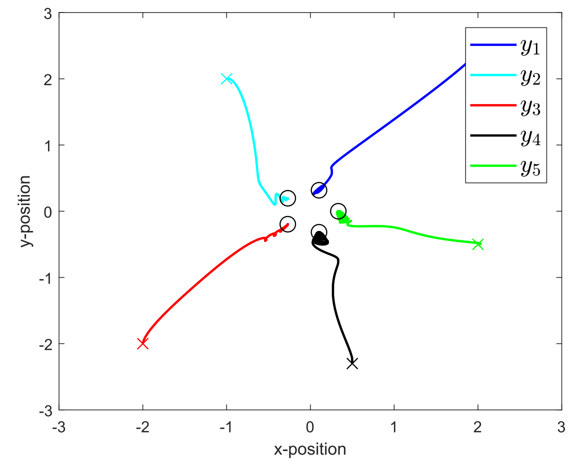

Fig. 8. Profiles of $y_i(t)$ under the controller (15) and (21) in *Example 2*.

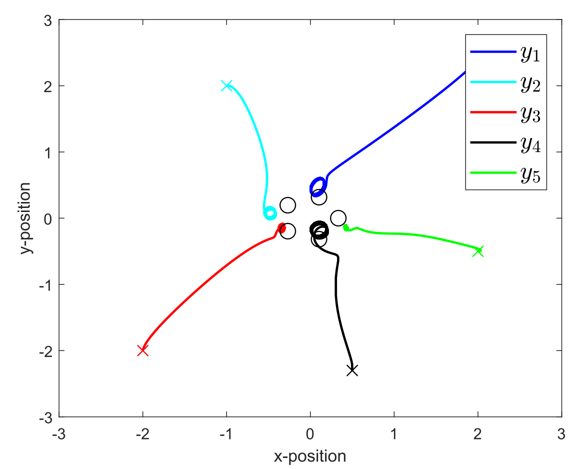

Fig. 9. Profiles of $y_i(t)$ under the controller (15) and (22) in *Example 2*.

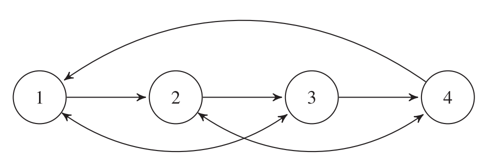

Fig. 10. Communication digraph $\mathcal{G}$ in *Example 3*.

weights depicted as Fig. 7 . The robots are designed to seek and surround a source of some signal with unknown distribution and keep close to each other for preserving connectivity.

Similar as in [2] and [18], we suppose the unknown distribution is quadratic as $f(s) = f^* + \frac{1}{2}||s - s^*||^2$ with source at $s^* \in \mathbb{R}^2$. Choose a cost function J_i for robot i as $J_i(y_i,\, y_{-i}) = ||y_i - s^* - r_i||^2 + \lambda \sum_{j=1}^{5} ||y_i - y_j||^2$ with r_i a translation vector for surrounding and a regularized parameter $\lambda > 0$ reflecting the importance of connectivity. Then, these mobile robots play a noncooperative game. It has a unique Nash equilibrium at $y^* = \mathrm{col}(y_1^*,\, \ldots,\, y_5^*)$ with $y_i^* = \frac{6\, s^* + r_i + \sum_{j=1}^{5} r_j}{5\lambda + 1}$.

Since the distribution f (or equivalently, the source location s^*) is unknown *a priori*, the generator (4) is not implementable. Nonetheless, we still can measure or learn the real-time gradient $\nabla f(y_i) = y_i - s^*$ of the distribution by onboard sensors at each position y_i during the convergence process. Then, we can compute the real-time partial gradient $\nabla_i J_i(y_i,\, \mathbf{z}_{-i}^i)$ and use algorithm (17) to solve this problem.

Moreover, agent i is supposed to have a nonconstant actuating disturbance modeled by $d_i(t) = D_i v_i(t)$, $\dot{v}_i = S_i v_i$ with

$$D_i = \begin{bmatrix} 1+\mu_1 & 1+\mu_2 & \mu_3 \\ 1+\mu_4 & \mu_5 & 1+\mu_6 \end{bmatrix}, \quad S_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{bmatrix}$$

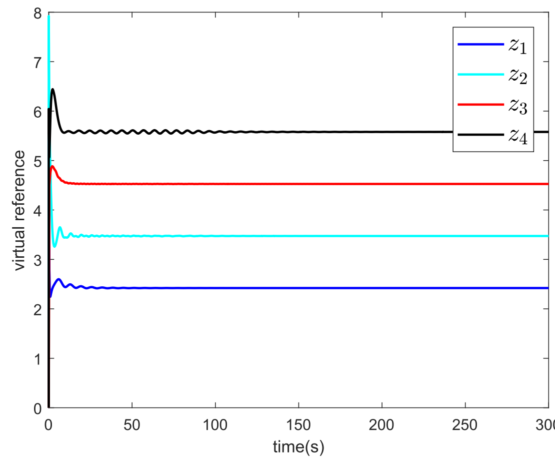

Fig. 11. Profiles of $z_i(t)$ under the controller (17) in *Example 3*.

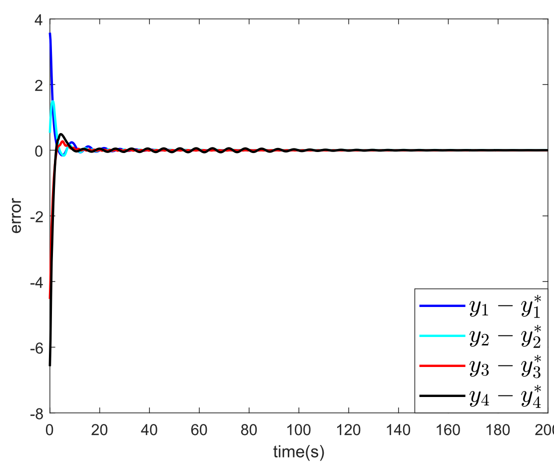

Fig. 12. Profiles of $y_i(t) - y_i^*$ under the controller (17) in *Example 3*.

where $\boldsymbol{\mu} = \mathrm{col}(\mu_1, \ldots, \mu_6) \in \mathbb{R}^6$ is an uncertain parameter vector satisfying $|\mu_i| \leq 0.2$. Although the pair (D, S) is verified to be observable, the observer-based approach proposed in [18] fails to solve this problem due to the uncertain parameter $\boldsymbol{\mu}$. Next, we show how to solve it by choosing a distributed controller of the form (2) for each input channel of (20).

Note that the disturbance can be represented as $d_i(t) = A_{0i} + A_{1i}\sin(i\,t) + A_{2i}\cos(i\,t)$ for some constant vectors $A_{0i}, A_{1i}, A_{2i} \in \mathbb{R}^2$ depending upon the initial value $v_i(0)$ and matrix D. Then, agent (20) can be rewritten into the form (1) with $n_i = 2$, $x_{1,i} = y_i$, $x_{2,i} = \dot{y}_i$, $\boldsymbol{p}_i(x_i, t) = \mathrm{col}(1, \sin(i\,t), \cos(i\,t))$, and $\theta_i = \mathrm{col}(A_{0i}^\intercal, A_{1i}^\intercal, A_{2i}^\intercal) \in \mathbb{R}^{3\times 2}$. Thus, we present the following tracking controller for agent i:

$$
\begin{aligned}
u_i &= -\hat{\theta}_i^\intercal \boldsymbol{p}_i(x_i, t) - 4(x_{1,i} - z_{1,i}) - 4x_{2,i}\\
\dot{\hat{\theta}}_i &= 5\boldsymbol{p}_i(x_i, t)\left[\frac{1}{4}(x_{1,i} - z_i) + \frac{5}{16}x_{2,i}\right]^\intercal.
\end{aligned}
\tag{21}
$$

For simulations, we suppose that the source is at the origin and $\lambda = 1$. Set $r_i = \mathrm{col}(\cos(i\omega^*), \sin(i\omega^*))$ with $\omega^* = \frac{2\pi}{5}$. Choose the initial conditions randomly and mark the start position of each robot by crosses. The evolution of robots' positions is shown in Fig. 8. One can find that the robots finally reach the Nash equilibrium position of the associated noncooperative game marked by circles. In comparison, we remove the adaption component in (21) and use the following static controller for agent i:

$$
u_i = -4(x_{1,i} - z_{1,i}) - 4x_{2,i}. \tag{22}
$$

Note that the closed-loop system under this controller is input-to-state stable with respect to the actuating disturbance as the input. Then, the output of agents will finally enter into a neighborhood of the Nash equilibrium, whose size depends on the strength of the disturbance. Moreover, these outputs cannot converge to the expected position, as shown in Fig. 9. These observations verify the effectiveness of our controller in handing unknown external disturbances.

Example 3: Consider a multiagent system including four controlled Van der Pol systems as follows:

$$
\begin{aligned}
\dot{x}_{1,i} &= x_{2,i}\\
\dot{x}_{2,i} &= -a_i x_{1,i} + b_i\left(1 - x_{1,i}^2\right)x_{2,i} + u_i\\
y_i &= x_{1,i}, \quad i = 1, 2, 3, 4
\end{aligned}
$$

where a_i, b_i are unknown positive constants. The information sharing graph of this multiagent system is depicted in Fig. 10 with unity edge weights.

We consider the Nash equilibrium seeking problem for this multiagent system with a local cost function $J_i(y_i, y_{-i}) = (y_i - y_{i0})^2 - y_i(p_i\sum_{i=1}^4 y_i + q_i)$ for agent i with $i = 1, 2, 3, 4$. Note that all these agents have unknown nonlinear dynamics. To make it more interesting, we further assume that agent i has an actuating disturbance $d_i(t)$ as in Example 2 described by

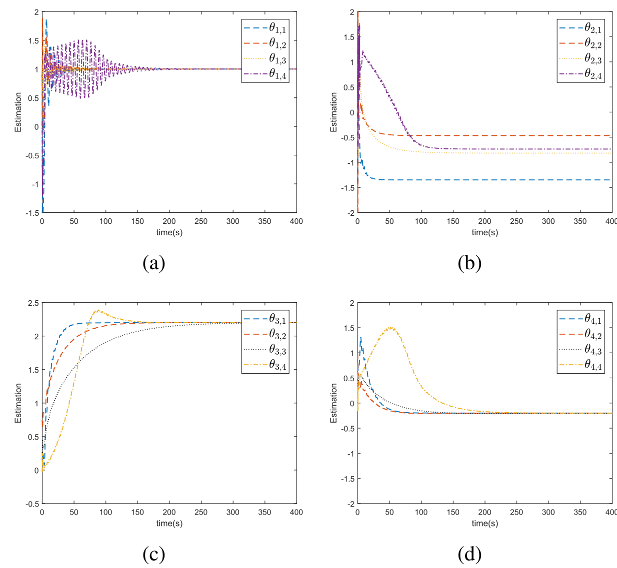

Fig. 13. Profiles of $\hat{\theta}_{i,j}(t)$ under the controller (17) in *Example 3*.

different system matrices

$$D_i = \begin{bmatrix} 1+\mu_1 & \mu_2 \end{bmatrix}, \quad S_i = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

with uncertain parameters $|\mu_1| \leq 0.5$ and $|\mu_2| \leq 0.5$.

Denote $\Delta_i(x_i,\, t) = -a_i x_{1,i} + b_i(1 - x_{1,i}^2)x_{2,i} + d_i(t)$. Note that $d_i(t) = A_{1i}\sin(i\, t) + A_{2i}\cos(i\, t)$ for some constants A_{1i}, A_{2i} depending upon the initial value $v_i(0)$ and D_i. We let

$$\boldsymbol{p}_i(x_i,\, t) = \text{col}\left(-x_{1,i},\, (1 - x_{1,i}^2)x_{2,i},\, \sin(i\, t),\, \cos(i\, t)\right)$$

$$\theta_i = \text{col}(\theta_{1,i},\, \ldots,\, \theta_{4,i}) = \text{col}(a_i,\, b_i,\, A_{1i},\, A_{2i}).$$

Then, these agents have been put into the form (1) with basis function vector $\boldsymbol{p}_i(x_i,\, t)$ defined as above and an unknown parameter vector $\theta_i \in \mathbb{R}^4$.

We let $p_i = 0.1$, $q_i = 1$, $y_{i0} = i$ and set the system parameters in agents as $a_i = b_i = 1$, $\mu_1 = 0.1$, $\mu_2 = -0.1$, $v_i(0) = \text{col}(0,\, 2)$ for $i = 1,\, \ldots,\, 4$. Assumptions 1–4 can be practically verified. Moreover, the Nash equilibrium of this noncooperative game is $y^* = \text{col}(2.42,\, 3.47,\, 4.53,\, 5.58)$ by numerical computations. According to Theorems 1 and 2, the Nash equilibrium seeking problem for these agents can be solved by different kinds of distributed controllers of the form (2). For simulations, we use the controller (17) using only real-time gradients. Choose $\alpha = 4$ for the virtual game dynamics (4) and $k_{1i} = -4$, $k_{2i} = -4$, $\Lambda_i = 5I_4$, $\epsilon = 0.8$ for the adaptive tracking controller. All initials are randomly chosen. Applying controller (17) to agent (1), the profiles of $z_i(t)$ and regulation error $y_i(t) - y_i^*$ are shown in Figs. 11 and 12. It can be found that the Nash equilibrium y^* is quickly reproduced even with real-time gradients while the convergence error $y_i(t) - y_i^*$ vanishes irrespective of the unknown nonlinearity Δ_i and external disturbance $d_i(t)$.

To explore the parameter convergence issue, we resort to Theorem 3 and Remark 9 and conclude that the estimators $\hat{\theta}_{1,i}$, $\hat{\theta}_{3,i}$, $\hat{\theta}_{4,i}$ will converge to their true values, while $\hat{\theta}_{2,i}$ may fail. These conclusions can be confirmed by Fig. 13.

VI. Conclusion

In this article, a Nash equilibrium seeking problem has been investigated for a typical class of high-order nonlinear systems with unknown dynamics. Following an embedded control procedure, we have developed a distributed adaptive controller to solve this problem under standard assumptions. The parameter convergence issue has also been addressed under some PE conditions. The algorithms show that we can achieve a co-design of decision making module, adaptive parameter estimating module, and tracking controller for general cyber-physical networks. Output feedback and coupling constraints for more general agent dynamics will be considered in our future work.

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