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Neural-network-based constrained optimal coordination for heterogeneous uncertain nonlinear multi-agent systems

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Abstract

In this article, we investigate a constrained optimal coordination problem for a class of heterogeneous nonlinear multi-agent systems described by high-order dynamics subject to both unknown nonlinearities and external disturbances. Each agent has a private objective function and a steady-state constraint about its output. We develop a composite distributed controller for each agent by a combination of internal model and neural network techniques. All agent outputs are proven to reach the constrained minimal point of the aggregate objective function with bounded residual errors irrespective of the unknown nonlinearities and external disturbances. Two examples are finally given to demonstrate the effectiveness of the algorithm.

KEYWORDS

multi-agent system, neural network, nonlinear control, optimal coordination

1 | INTRODUCTION

Multi-agent coordination has been a hot topic over the last decades and has many practical applications in multi-robot control, smart grid, and sensor networks.[1-3] As one of the most interesting problems, distributed consensus optimization attracts more and more attention due to the fast development of machine learning and big data technologies. Various effective algorithms have been proposed to achieve such an optimal coordination in different situations.[4-6]

Recently, many efforts have been made to incorporate high-order agent dynamics into the distributed optimization design. This is mainly due to the fact that distributed optimization tasks may be implemented or depend on physical plants of high-order dynamics in practice, for example, source seeking in multi-robot systems[7] and attitude formation control of rigid bodies.[8] For example, an optimal coordination problem was discussed in Zhang and Ordóñz[9] via an integral control rule for both double integrators and Euler–Lagrange agents. Distributed optimization with bounded controls was also explored for both single and double integrators in Xie and Lin.[10] For general linear systems, an embedded technique was developed in Tang et al.[11] to simplify the whole design by converting the original optimal coordination problem into several subproblems and solving them almost independently. At the same time, some interesting attempts have also been made for special classes of nonlinear multi-agent systems. For example, Wang et al.[12] focused on a class of nonlinear agents in output feedback form with unity relative degree and solved its optimal coordination problem by improving the integral rule in Zhang and Ordóñz.[9] The embedded design idea was also further exploited for nonlinear agents in different forms.[13-15]

So far, there are few optimal coordination results considering optimization constraints on the final states except for single-integrator multi-agent systems as mentioned above. Compared with the unconstrained case, the set constraint will pose some specific challenges. In fact, additional mechanisms are usually needed to ensure the satisfaction of constraints on decision variables.[16-19] Thus, the design of effective algorithms and the associated convergence analysis are more involved. When facing multi-agent systems of nontrivial dynamics, the problem will inevitably be much more challenging than the optimal coordination results derived either for single integrators or without such constraints.

Based on these observations, this article focuses on the optimal coordination problem for a typical class of heterogeneous nonlinear multi-agent systems with set constraints. Moreover, we assume the high-order agents are subject to both unknown nonlinearities and external disturbances. To overcome the difficulties brought by the nonlinearity, uncertainty, and constrained optimization requirement, we view the formulated problem as an asymptotic regulation problem where the reference point is determined by the constrained optimization problem and develop a novel neural-network-based distributed control to solve the optimal coordination problem. We also provide rigorous theoretical analysis to ensure the global stability of resultant closed-loop systems. To our knowledge, this might be the first attempt to solve such kind of optimal coordination problems by neural-network-based controls in this setting.

The main contributions of this article are twofold. On the one hand, we present and solve a constrained optimal coordination problem for high-order nonlinear agents. Compared with existing optimal coordination results for high-order agents,[9-11,13,14,20] this article extends them to the case when the global optimization problem has set constraints. On the other hand, a novel neural-network-based controller combined with internal model designs is developed to achieve the optimal coordination goal in a distributed manner. The developed controller can reject external disturbances generated by certain autonomous linear exosystem while only the disturbance-free cases were discussed in some existing works.[10,11,13,14,20] More importantly, thanks to the approximation ability of neural networks, the composite design method allows us to handle a large class of nonlinear high-order agents with uncertain dynamics. In fact, by removing the restrictive linearly parameterized condition on unknown nonlinearities, this work explicitly generalizes existing results for either linear or nonlinear multi-agent systems.[11,13]

The remainder of this article is organized as follows: Problem formulation is presented in Section 2. Then the main result is provided in Section 3 with detailed designs. Following that, two numerical examples are given to illustrate the efficiency of our algorithm in Section 4. Finally, conclusions are given in Section 5.

Notation: Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathbb{R}^{n\times m}$ be the set of all $n\times m$ matrices. 1_n (or 0_n) denotes an n-dimensional all-one (or all-zero) column vector and $\mathbf{1}_{n\times m}$ (or $\mathbf{0}_{n\times m}$) all-one (or all-zero) matrix. $\mathrm{diag}\{b_1, \dots, b_n\}$ denotes an $n\times n$ diagonal matrix with diagonal elements b_i, $(i = 1, \dots, n)$. $\mathrm{col}(a_1, \dots, a_n) = [a_1^{\intercal}, \dots, a_n^{\intercal}]^{\intercal}$ for column vectors a_i $(i = 1, \dots, n)$. For a vector x (or matrix A), $||x||$ ($||A||$) denotes its Euclidean (or spectral) norm. For a square matrix A, $\mathrm{Tr}(A)$ denotes the trace of A and $||A||_{\mathrm{F}} = \mathrm{Tr}(A^{\intercal}A)$ denotes its Frobenius norm. A continuous function $\alpha : [0, +\infty) \to [0, +\infty)$ belongs to class $\mathcal{K}_\infty$ if it is strictly increasing and satisfies $\alpha(0) = 0$ and $\lim_{s\to\infty}\alpha(s) = \infty$.

2 | PROBLEM FORMULATION

In this article, we consider a collection of N heterogeneous nonlinear systems modeled by:

$$x_i^{(n_i)} = g_i([x]_i,\ \mu) + b_i u_i + d_i(t), \quad i \in \mathcal{N} = \{1, \dots, N\}, \tag{1}$$

where $[x]_i \triangleq \mathrm{col}(x_i, \dots, x_i^{(n_i-1)}) \in \mathbb{R}^{qn_i}$ is the state variable of agent i with integer $n_i \geq 2$, x_i is the output, $u_i \in \mathbb{R}^q$ is the control input, and $\mu \in \mathbb{R}^{n_\mu}$ is an uncertain parameter vector. The high-frequency gain matrix b_i is invertible. Without loss of generality, we let $b_i = I_q$ and assume the vector-valued function $g_i : \mathbb{R}^{qn_i}\times\mathbb{R}^{n_\mu} \to \mathbb{R}^q$ to be smooth but unknown to us.

The signal $d_i(t) \in \mathbb{R}^q$ represents the external disturbance of agent i modeled by

$$d_i(t) = D_i(\mu)\omega_i, \quad \dot{\omega}_i = S_i\omega_i, \quad \omega_i(0) = \omega_{i0} \in \mathbb{R}^{m_i} \tag{2}$$

with $\omega_i \in \mathbb{R}^{m_i}$, $S_i \in \mathbb{R}^{m_i\times m_i}$ and $D_i \in \mathbb{R}^{q\times m_i}$. Moreover, we assume that the matrix S_i has no eigenvalue with negative real part. Note that system (2) can model many typical disturbances, including a combination of step signals of arbitrary magnitudes, ramp signals of arbitrary slopes, and sinusoidal signals of arbitrary amplitudes and initial phases.[21]

As stated in existing publications,[9,10,22] we endow this multi-agent system with the following distributed optimization problem

$$\begin{aligned} &\text{minimize} \quad f(y) = \sum_{i=1}^{N} f_i(y), \\ &\text{subject to} \quad y \in \Omega_0 \triangleq \bigcap_{i=1}^{N} \Omega_i \subset \mathbb{R}^q, \end{aligned} \tag{3}$$

where $f_i : \mathbb{R}^q \to \mathbb{R}$ is differentiable. Furthermore, we assume that each agent only know a part of this optimization problem in the sense that agent i only knows f_i and Ω_i.

To ensure the well-posedness of this optimization problem, we make the following assumption.[23]

Assumption 1. For $i \in \mathcal{N}$, the set Ω_i is closed and convex with Ω_0 nonempty; the function f_i is $\underline{l}_i$-strongly convex and its gradient ∇f_i is $\bar{l}_i$-Lipschitz over an open set containing Ω_i for constants $\underline{l}_i,\ \bar{l}_i > 0$.

Under this assumption, there exists a unique finite solution to problem (3) according to Theorem 2.2.10 in Nesterov.[23] Denote it by $y^* = \operatorname{argmin}_{y\in\Omega_0} f(y)$. We aim to regulate the multi-agent system (1) such that the agent outputs reach this global minimizer in spite of the uncertainties and disturbances. However, no agent can compute the exact y^* and reach it as expected by itself due to the lack of global information of f and Ω_0. To see this point, we can introduce some local decision variables for the agents as $y_1, \ \dots \ , y_N$ and let $\tilde{f}(\boldsymbol{y}) = \sum_{i=1}^{N} f_i(y_i)$ with $\boldsymbol{y} \triangleq \operatorname{col}(y_1, \ \dots \ , y_N)$. Then, the problem (3) is equivalent to minimize $\tilde{f}(\boldsymbol{y}) = \sum_{i=1}^{N} f_i(y_i)$ subject to a local set constraint $y_i \in \Omega_i$ and a global consensus constraint $y_1 = \cdots = y_N$. Since the consensus constraint can only be satisfied via a cooperation, no agent can thus reach the expected optimal solution y^* by itself. In this article, we are more interested in distributed designs where the agents can communicate with some others.

To this end, we use a directed graph $\mathcal{G} = (\mathcal{N},\ \mathcal{E},\ \mathcal{A})$ to describe the information-sharing relationships among those agents with a node set $\mathcal{N}$, an edge set $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$, and a weight matrix $\mathcal{A} \in \mathbb{R}^{N\times N}$.[24] If agent i can get the information of agent j, then there is an edge $(j,\ i)$ in $\mathcal{E}$, that is, $a_{ij} > 0$. Here is an assumption to guarantee that any agent's information can reach another.

Assumption 2. Graph $\mathcal{G}$ is undirected and connected.

Regarding multi-agent system (1), function f_i, set Ω_i, and graph $\mathcal{G}$, the constrained optimal coordination problem for agent (1) is formulated to find a feedback control u_i for agent i by using its own and exchanged information with the neighbors such that all trajectories of agents are well-defined over the time interval $[0,\ +\infty)$ and the resultant outputs satisfy $\lim_{t\to+\infty} \|x_i(t) - y^*\| = 0$ for each $i \in \mathcal{N}$.

This optimal coordination problem naturally ensures an output consensus of the multi-agent system (1). Compared with existing output consensus coordination results,[25-27] the formulation further requires the consensus point to be the optimal solution y^* specified by minimizing a global cost function across the whole network, which is more challenging. It is remarkable that the optimality issue of multi-agent coordination has also been studied from the viewpoint of optimal control.[28-30] Different from these important results, we emphasize more on the optimal steady-state performance and require the agent outputs to reach a consensus and minimize some global static optimization problem.

The formulated optimal coordination problem has been partially investigated for second-order agents.[9,10,20] Here, we further consider heterogeneous set constraints and higher-order agent dynamics possessing unknown nonlinearities and external disturbances, which inevitability bring more difficulties in achieving such an optimal coordination than these existing results.

3 | MAIN RESULT

To solve this problem, we develop an internal model + neural network based design. The full design scheme is illustrated in Figure 1. Briefly, we will first consider an auxiliary optimal coordination problem for certain virtual multi-agent systems and convert the original problem into a robust stabilization problem. Then, we use internal models to reject the external disturbance and neural network to approximate the uncertain nonlinearity. These two components together with an adaptive stabilizer finally solve the formulated optimal coordination problem for the physical multi-agent system (1).

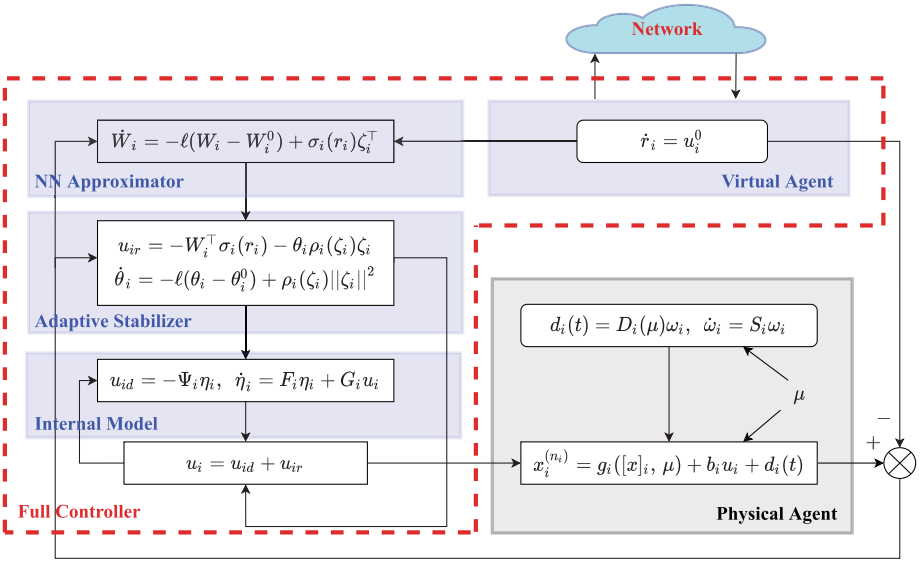

FIGURE 1 Block diagram to illustrate the design procedure

3.1 | Problem conversion

We start from the same optimal coordination problem for a group of virtual integrator agents as follows:

$$\dot{r}_i = u_i^0 \tag{4}$$

with state $r_i \in \mathbb{R}^q$ and input $u_i^0 \in \mathbb{R}^q$. Assign these agents with the same cost functions $f_1, \ \dots \ , f_N$ and graph $\mathcal{G}$ as above. To solve the optimal coordination problem for agent (4) is to develop proper input u_i^0 for agent i such that $\lim_{t\to+\infty} r_i(t) = y^*$. It is interesting to remark that the virtual agent (4) can be taken as a reference leader for the original agent i. In fact, if this auxiliary problem can be solved, we only need to drive the original agents to track the outputs of these virtual agents to achieve the expected optimal coordination for agent (1). Similar ideas can be found in many existing multi-agent coordination papers.[25,31-34]

Note that this auxiliary problem has been well-studied in literature and many distributed algorithms can be utilized to solve it, for example, the ones in Liu and Wang[35] and Zeng et al..[36] We suppose this auxiliary optimal coordination problem has been done. With these estimates of the global optimal solution y^* given by (4), we only need to consider a robust tracking problem for agent (1) with reference $r_i(t)$ to solve our formulated problem. For better analysis, we denote $x_{i1} = x_i - r_i$, $x_{ij} \triangleq x_i^{(j-1)}$ for $2 \le j \le n_i$. Choose constants k_{ij} for $1 \le j \le n_i - 1$ such that the polynomial $p_i(\lambda) = \sum_{j=1}^{n_i-1} k_{ij}\lambda^{j-1} + \lambda^{n_i-1}$ is Hurwitz. Letting $z_i = \text{col}(x_{i1}, \ \dots \ , x_{in_i-1})$ and $\zeta_i = \sum_{j=1}^{n_i-1} k_{ij}x_{ij} + x_{in_i}$ gives an error system as follows:

$$\begin{aligned} \dot{z}_i &= A_i z_i + B_i\zeta_i - E_i\dot{r}_i, \\ \dot{\zeta}_i &= \bar{g}_i(z_i, \ \zeta_i, \ r_i, \ \mu) + u_i + d_i(t) - k_{i1}\dot{r}_i, \end{aligned} \tag{5}$$

where $\bar{g}_i(z_i, \ \zeta_i, \ r_i, \ \mu) = g_i([x]_i, \ \mu) + k_{in_i-1}\zeta_i - k_{in_i-1}k_{i1}x_{i1} + \sum_{j=2}^{n_i-1}(k_{ij-1} - k_{in_i-1}k_{ij})x_{ij}$ and

$$A_i = \left[\begin{array}{c|c} \mathbf{0}_{n_i-2} & I_{n_i-2} \\ \hline -k_{i1} & -k_{i2}, \dots, -k_{in_i-1} \end{array}\right] \otimes I_q, \quad B_i = \begin{bmatrix} \mathbf{0}_{n_i-2} \\ 1 \end{bmatrix} \otimes I_q, \quad E_i = \begin{bmatrix} 1 \\ \mathbf{0}_{n_i-2} \end{bmatrix} \otimes I_q.$$

From the above form, we have converted the formulated optimal coordination problem into some robust stabilization problem by viewing $\dot{r}_i$ as a vanishing perturbation. Compared with similar results for linear systems,[10,11,20] our problem involves extra nonlinearities from the set constraints and nonlinear agent dynamics. Moreover, the nonlinearities in this multi-agent system cannot be perfectly linearly parameterized as that in Tang.[13] Consequently, (adaptive) feedback

linearization method is not applicable to the associated tracking problem because of the unknown nonlinearity and external disturbances. Then, we have to seek new rules to solve our problem.

Inspired by existing designs,[14,37,38] we split the whole control effort into two parts as follows:

$$u_i = u_{id} + u_{ir}, \tag{6}$$

where u_{id} is designed to compensate the external disturbance and u_{ir} is to handle the unknown nonlinearity and drive agent i to track its reference r_i.

It is well-known that internal-model-based control is effective to reject modeled disturbances.[21] Here, we construct u_{id} following the same technical line. Let $P_i(s) = s^{n_{p_i}} + \hat{p}_{i1}s^{n_{p_i}-1} + \cdots + \hat{p}_{in_{p_i}-1}s + \hat{p}_{in_{p_i}}$ be the minimal polynomial of matrix S_i and denote $\tau_i = \text{col}(\tau_{i1}, \ldots, \tau_{in_{p_i}})$ with $\tau_{ij} = \frac{\mathrm{d}^{j-1}d_i(t)}{\mathrm{d}t^{j-1}} \in \mathbb{R}^q$. Take two matrices as follows:

$$\Phi_i = \left[\begin{array}{c|c} 0 & I_{n_{p_i}-1} \\ \hline -\hat{p}_{in_{p_i}} & -\hat{p}_{in_{p_i}-1} \cdots -\hat{p}_{i1} \end{array}\right] \otimes I_q, \quad \Psi_i = \begin{bmatrix} 1 \\ \mathbf{0}_{n_{pi}-1} \end{bmatrix}^{\top} \otimes I_q.$$

By a direct calculation, we obtain

$$\dot{\tau}_i = \Phi_i\tau_i, \quad d_i = \Psi_i\tau_i. \tag{7}$$

System (7) is called a steady-state generator.[21] Since the pair (Ψ_i, Φ_i) is observable, there exists a constant matrix G_i such that $F_i \triangleq \Phi_i + G_i\Psi_i$ is Hurwitz. To reject the disturbance d_i, we propose an internal-model-based compensator

$$u_{id} = -\Psi_i\eta_i, \quad \dot{\eta}_i = F_i\eta_i + G_iu_i. \tag{8}$$

Next, we are going to propose applicable u_{ir} to complete the whole design. Since the nonlinear function $g_i([x]_i, \mu)$ is unknown to us, the term $\bar{g}_i(z_i, \zeta_i, r_i, \mu)$ cannot be directly used for feedback. To tackle this issue, an intuitive idea is to estimate this term in some way and develop an estimation-based control law. As neural networks have been proven to be an effective tool to approximate unknown nonlinear functions,[39-42] we present a neural-network-based rule combined with the above internal-model-based compensator to solve our problem in the next subsection.

3.2 | Solvability analysis

In this subsection, we present the whole design of our optimal coordination rule and provide theoretical stability analysis of the closed-loop system.

For our optimal coordination problem, we aim at global stability performance. Note that neural-network-based controls usually ensure control performance in the sense of semiglobal stability of the closed-loop systems.[40,41] To overcome this shortcoming, we try to utilize the neural networks to estimate the expected feedforwarding control efforts as that in Chen et al..[43] For this purpose, we let $\boldsymbol{u}_i(y^*, \mu) = g_i(\text{col}(y^*, \mathbf{0}_{q(n_i-1)}), \mu)$. This is indeed the feedforwarding effort for us to regulate x_i to the optimal point according to Theorem 3.8 in Huang.[21] When the trajectory of $r_i(t)$ converges to the optimal solution y^*, it should be uniformly bounded. Thus, we try to reproduce $\boldsymbol{u}_i(r_i, \mu)$ and develop neural-network-based approximation rules for the sequel design.

To this end, we let $\hat{g}_i(z_i, \zeta_i, r_i, \mu) \triangleq \bar{g}_i(z_i, \zeta_i, r_i, \mu) - \boldsymbol{u}_i(r_i, \mu)$. It is verified that $\hat{g}_i(\mathbf{0}, \mathbf{0}, r_i, \mu) = \mathbf{0}$ for any $r_i \in \mathbb{R}^q$ and $\mu \in \mathbb{R}^{n_\mu}$. Motivated by existing neural-network-based designs,[37,43,44] we use a radial basis function (RBF) network as a function approximator and rewrite $\boldsymbol{u}_i$ as follows:

$$\boldsymbol{u}_i(r_i, \mu) = W_i^{\top}\boldsymbol{\sigma}_i(r_i) + \boldsymbol{\epsilon}_i(r_i),$$

where $\boldsymbol{\sigma}_i(r_i) = \text{col}(\sigma_{i1}(r_i), \ldots, \sigma_{in_w}(r_i))$ is the activation function vector with $\sigma_{ij}(s) = e^{-(s-\mu_{ij}^c)^2/\kappa_i^2}$ for $j = 1, \ldots, n_w$ and $W_i \in \mathbb{R}^{n_w \times q}$ is the weight matrix. Here, μ_{ij}^c is the center of the receptive field, κ_i is the width of the Gaussian function, and $\boldsymbol{\epsilon}_i(r_i)$ is the residual error. By the universal approximation theorem,[40] for any given $\varepsilon > 0$, there exists an ideal constant weight $W_i^* \in \mathbb{R}^{n_w \times q}$ with a large enough integer $n_w > 0$ such that $||\boldsymbol{\epsilon}_i(r_i)|| < \varepsilon$ over any compact set.

Since the ideal weight W_i^* cannot be known a prior, we develop the following adaptive neural-network-based rule to tackle this issue:

$$\begin{aligned} u_{ir} &= -W_i^{\intercal}\boldsymbol{\sigma}_i(r_i) - \theta_i\rho_i(\zeta_i)\zeta_i, \\ \dot{W}_i &= -\ell(W_i - W_i^0) + \boldsymbol{\sigma}_i(r_i)\zeta_i^{\intercal}, \\ \dot{\theta}_i &= -\ell(\theta_i - \theta_i^0) + \rho_i(\zeta_i)||\zeta_i||^2, \end{aligned} \tag{9}$$

where function $\rho_i > 0$ is to be specified later. Here W_i, θ_i are dynamic gains, $\ell > 0$ is a fixed chosen constant to ensure the boundedness of W_i and θ_i, and the term $-\theta_i\rho_i(\zeta_i)\zeta_i$ is designed to dominate the unknown nonlinearity in (5). Similar adaptive controllers have been used in literature.[41,44] The constants W_i^0 and θ_i^0 are chosen parameters based on the (possible) prior information of this multi-agent system, especially the nonlinearities and initial conditions of the whole system. Without further requirements, we can just set the default values as $W_i^0 = \mathbf{0}$ and $\theta_i^0 = 0$.

For the auxiliary constrained optimal coordination problem, we directly borrow the cooperative laws developed in Liu and Wang.[35] Combining (8) and (9), we propose the full optimal coordination controller for agent i as follows:

$$\begin{aligned} u_i &= -W_i^{\intercal}\boldsymbol{\sigma}(r_i) - \theta_i\rho_i(\zeta_i)\zeta_i - \Psi_i\eta_i, \\ \dot{\eta}_i &= F_i\eta_i + G_iu_i, \\ \dot{W}_i &= -\ell(W_i - W_i^0) + \boldsymbol{\sigma}_i(r_i)\zeta_i^{\intercal}, \\ \dot{\theta}_i &= -\ell(\theta_i - \theta_i^0) + \rho_i(\zeta_i)||\zeta_i||^2, \\ \dot{r}_i &= -2r_i + 2P_{\Omega_i}(r_i - \nabla f_i(r_i) - r_i^{\boldsymbol{o}} - v_i^{\boldsymbol{o}}), \\ \dot{v}_i &= r_i, \end{aligned} \tag{10}$$

where $r_i^{\boldsymbol{o}} \triangleq \sum_{j=1}^N a_{ij}(r_i - r_j)$, $v_i^{\boldsymbol{o}} \triangleq \sum_{j=1}^N a_{ij}(v_i - v_j)$, and P_{Ω_i} is the projector operator from $\mathbb{R}^q$ to Ω_i. Here, the η_i subsystem is the internal model to reject the modeled disturbance (2), W_i, θ_i are adaptive parameters in the neural networks to approximate the expected feedforwarding control efforts, and r_i, v_i are utilized to generate the global optimal solution y^*. Clearly, this controller is distributed as agent i only uses its own and neighboring information.

Putting nonlinear agent (1) and distributed controller (10) together, we obtain the associated closed-loop system as follows:

$$\begin{aligned} \dot{z}_i &= A_iz_i + B_i\zeta_i - E_i\dot{r}_i, \\ \dot{\zeta}_i &= \hat{g}_i(z_i,\ \zeta_i,\ r_i,\ \mu) + \boldsymbol{u}_i(r_i,\ \mu) - W_i^{\intercal}\boldsymbol{\sigma}(r_i) - \theta_i\rho_i(\zeta_i)\zeta_i - \Psi_i\eta_i + d_i - k_{i1}\dot{r}_i, \\ \dot{\eta}_i &= F_i\eta_i + G_iu_i, \\ \dot{W}_i &= -\ell(W_i - W_i^0) + \boldsymbol{\sigma}_i(r_i)\zeta_i^{\intercal}, \\ \dot{\theta}_i &= -\ell(\theta_i - \theta_i^0) + \rho_i(\zeta_i)||\zeta_i||^2, \\ \dot{r}_i &= -2r_i + 2P_{\Omega_i}(r_i - \nabla f_i(r_i) - r_i^{\boldsymbol{o}} - v_i^{\boldsymbol{o}}), \\ \dot{v}_i &= r_i. \end{aligned} \tag{11}$$

It is ready to provide our main theorem of this article.

Theorem 1. *Consider the multi-agent system (1) with graph $\mathcal{G}$ and function f_i and suppose Assumptions 1 and 2 holds. Using controller (10) to solve the constrained optimal coordination problem, one has the following results:*

- *The trajectories of $x_i(t),\ \dots\ , x_i^{(n_i-1)}(t)$ are bounded for all $t \geq 0$, $i = 1,\ \dots\ , N$;*
- *The coordination errors $||x_i(t) - y^*||$ are uniformly ultimately bounded, that is, this multi-agent system achieves approximate optimal coordination with residual errors.*

Proof. Let $\overline{\eta}_i = \eta_i - \tau_i - G_i\zeta_i$, $\overline{W}_i = W_i - W_i^*$, and $\overline{\theta}_i = \theta_i - \theta_i^*$ with $\theta_i^* > 0$ to be specified later. Then, we have

$$\begin{aligned} \dot{z}_i &= A_iz_i + B_i\zeta_i - E_i\dot{r}_i, \\ \dot{\overline{\eta}}_i &= F_i\overline{\eta}_i + F_iG_i\zeta_i - G_i\hat{g}_i(z_i,\ \zeta_i,\ r_i,\ \mu) + k_{i1}\dot{r}_i, \end{aligned}$$

$$
\begin{aligned}
\dot{\zeta}_i &= \tilde{g}_i(z_i,\ \overline{\eta}_i,\ \zeta_i,\ r_i,\ \mu) - \overline{W}_i^{\intercal}\boldsymbol{\sigma}(r_i) - \theta_i^* \rho_i(\zeta_i)\zeta_i - \overline{\theta}_i \rho_i(\zeta_i)\zeta_i + \epsilon_i(r_i) - k_{i1}\dot{r}_i,\\
\dot{\overline{W}}_i &= -\ell(\overline{W}_i + W_i^* - W_i^0) + \boldsymbol{\sigma}_i(r_i)\zeta_i^{\intercal},\\
\dot{\overline{\theta}}_i &= -\ell(\overline{\theta}_i + \theta_i^* - \theta_i^0) + \rho_i(\zeta_i)||\zeta_i||^2,\\
\dot{\boldsymbol{r}} &= -2\boldsymbol{r} + 2P_{\Omega}(\boldsymbol{r} - \nabla\tilde{f}(\boldsymbol{r})) - (L\otimes I_q)\boldsymbol{r} - (L\otimes I_q)\boldsymbol{v},\\
\dot{\boldsymbol{v}} &= \boldsymbol{r},
\end{aligned}
\tag{12}
$$

where $\boldsymbol{r} = \mathrm{col}(r_1,\ \dots,\ r_N)$, $\boldsymbol{v} = \mathrm{col}(v_1,\ \dots,\ v_N)$, P_{Ω} is the projector operator determined by P_{Ω_i}, and $\tilde{g}_i(z_i,\ \overline{\eta}_i,\ \zeta_i,\ r_i,\ \mu) = \hat{g}_i(z_i,\ \zeta_i,\ r_i,\ \mu) - \Psi_i\overline{\eta}_i + \Psi_i G_i\zeta_i$. It can be easily verified that $\bar{g}_i(\mathbf{0},\ \mathbf{0},\ r_i,\ \mu) = \mathbf{0}$ and $\tilde{g}_i(\mathbf{0},\ \mathbf{0},\ \mathbf{0},\ r_i,\ \mu) = 0$ for any $r_i \in \mathbb{R}^q$ and $\mu \in \mathbb{R}^{n_\mu}$. The proof can be split into two steps as follows:

Step 1: We consider the stability of the first two subsystems $\mathrm{col}(z_i, \overline{\eta}_i)$. As the matrices A_i and F_i are Hurwitz, there exist unique positive definite matrices P_{iz} and $P_{i\eta}$ such that $P_{iz}A_i + A_i^{\intercal}P_{iz} = -2I_{q(n_i-1)}$ and $P_{i\eta}F_i + F_i^{\intercal}P_{i\eta} = -2I_{qn_{p_i}}$. Letting $V_{iz} = z_i^{\intercal}P_{iz}z_i$ and $V_{i\overline{\eta}} = \overline{\eta}_i^{\intercal}P_{i\overline{\eta}}\overline{\eta}_i$ gives

$$
\begin{aligned}
\dot{V}_{iz} &= 2z_i^{\intercal}P_{iz}(A_iz_i + B_i\zeta_i - E_i\dot{r}_i)\\
&= 2z_i^{\intercal}P_{iz}A_iz_i + 2z_i^{\intercal}P_{iz}B_i\zeta_i - 2z_i^{\intercal}P_{iz}E_i\dot{r}_i\\
&\le -||z_i||^2 + 2||P_{iz}B_i||^2||\zeta_i||^2 + 2||P_{iz}E_i||^2||\dot{r}_i||^2
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{V}_{i\overline{\eta}} &= 2\overline{\eta}_i^{\intercal}P_{i\eta}[F_i\overline{\eta}_i + F_iG_i\zeta_i - G_i\hat{g}_i(z_i,\ \zeta_i,\ r_i,\ \mu) + k_{i1}\dot{r}_i]\\
&= 2\overline{\eta}_i^{\intercal}P_{i\eta}F_i\overline{\eta}_i + 2\overline{\eta}_i^{\intercal}P_{i\eta}F_iG_i\zeta_i - 2\overline{\eta}_i^{\intercal}P_{i\eta}G_i\hat{g}_i(z_i,\ \zeta_i,\ r_i,\ \mu) + 2\overline{\eta}_i^{\intercal}P_{i\eta}k_{i1}\dot{r}_i\\
&\le -||\overline{\eta}_i||^2 + 3||P_{i\eta}F_iG_i||^2||\zeta_i||^2 + 3||P_{i\eta}k_{i1}||^2||\dot{r}_i||^2\\
&\quad + 3||P_{i\eta}G_i||^2||\hat{g}_i(z_i,\ \zeta_i,\ r_i,\ \mu)||^2.
\end{aligned}
$$

Note that $\hat{g}_i(\mathbf{0},\ \mathbf{0},\ r_i,\ \mu) = 0$ and $\tilde{g}_i(\mathbf{0},\ \mathbf{0},\ \mathbf{0},\ r_i,\ \mu) = 0$ for any r_i and μ. By Lemma 7.8 in Huang,[21] there exist some known smooth functions $\hat{\phi}_{i1}, \hat{\phi}_{i2}, \tilde{\phi}_{i1},\ \tilde{\phi}_{i2} > 1$ and unknown constants $\hat{c}_{ig}, \tilde{c}_{ig} > 1$ such that

$$
\begin{aligned}
||\hat{g}_i(z_i,\ \zeta_i,\ r_i,\ \mu)||^2 &\le \hat{c}_{ig}[\hat{\phi}_{i1}(z_i)||\zeta_i||^2 + \hat{\phi}_{i2}(\zeta_i)||\zeta_i||^2],\\
||\tilde{g}_i(z_i,\ \overline{\eta}_i,\ \zeta_i,\ r_i,\ \mu)||^2 &\le \tilde{c}_{ig}[\tilde{\phi}_{i1}(\tilde{z}_i)||\tilde{z}_i||^2 + \tilde{\phi}_{i2}(\zeta_i)||\zeta_i||^2],
\end{aligned}
\tag{13}
$$

where $\tilde{z}_i \triangleq \mathrm{col}(z_i,\ \overline{\eta}_i)$ for short.

We apply Theorem 1 in Sontag and Teel[45] to z_i-subsystem and obtain that, for any given smooth $\Delta_{iz}(z_i) > 0$, there exists a differentiable function $V_{iz}^1(z_i)$ satisfying that

$$
\begin{aligned}
\underline{\alpha}_{iz}(||z_i||) \le V_{iz}^1(z_i) &\le \overline{\alpha}_{iz}(||z_i||)\\
\dot{V}_{iz}^1 &\le -\Delta_{iz}(z_i)||z_i||^2 + \sigma_{i\zeta}^1\gamma_{i\zeta}^1(\zeta_i)||\zeta_i||^2 + \sigma_{ir}^1\gamma_{ir}^1(\dot{r}_i)||\dot{r}_i||^2
\end{aligned}
$$

for some known smooth functions $\underline{\alpha}_{iz},\ \overline{\alpha}_{iz} \in \mathcal{K}_\infty, \gamma_{i\zeta}^1,\ \gamma_{ir}^1 \ge 1$ and unknown constants $\sigma_{i\zeta}^1,\ \sigma_{ir}^1 \ge 1$.

Let $V_{i\tilde{z}} = \ell_{iz}V_{iz}^1(z_i) + V_{i\overline{\eta}}(\overline{\eta}_i)$ with a constant $\ell_{iz} > 0$ to be specified later. It is positive definite and radially unbounded. Its time derivative along the trajectory of system (12) satisfies

$$
\begin{aligned}
\dot{V}_{i\tilde{z}} &\le -\ell_{iz}\Delta_{iz}(z_i)||z_i||^2 + \ell_{iz}\sigma_{i\zeta}^1\gamma_{i\zeta}^1(||\zeta_i||)||\zeta_i||^2 + \ell_{iz}\sigma_{ir}^1\gamma_{ir}(||\dot{r}_i||)||\dot{r}_i||^2 - ||\overline{\eta}_i||^2 + 3||P_{i\eta}F_iG_i||^2||\zeta_i||^2\\
&\quad + 3||P_{i\eta}G_i||^2||\hat{g}_i(z_i,\ \zeta_i,\ r_i,\ \mu)||^2 + 3||P_{i\eta}k_{i1}||^2||\dot{r}_i||^2\\
&\le -[\ell_{iz}\Delta_{iz}(z_i) - 3\hat{c}_{ig}||P_{i\eta}G_i||^2\hat{\phi}_{i1}(||z_i||)]||z_i||^2 - ||\overline{\eta}_i||^2 + [\ell_{iz}\sigma_{i\zeta}^1\gamma_{i\zeta}^1(\zeta_i) + 3||P_{i\eta}F_iG_i||^2\\
&\quad + 3\hat{c}_{ig}||P_{i\eta}G_i||^2\hat{\phi}_{i2}(\zeta_i)]||\zeta_i||^2 + [\ell_{iz}\sigma_{ir}^1\gamma_{ir}^1(\dot{r}_i) + 3||P_{i\eta}k_{i1}||^2]||\dot{r}_i||^2.
\end{aligned}
$$

Let $\Delta_{iz},\ \tilde{\gamma}_{i\zeta},\ \tilde{\gamma}_{ir}$ be smooth functions satisfying

$$
\Delta_{iz}(z_i) \ge 2\max\{\hat{\phi}_{i1}(||z_i||),\ 1\},\quad \tilde{\gamma}_{i\zeta}(\zeta_i) \ge 3\max\{\gamma_{i\zeta}^1(\zeta_i)\hat{\phi}_{i2}(\zeta_i),\ 1\},\quad \tilde{\gamma}_{ir}(\dot{r}_i) \ge 2\max\{\gamma_{ir}^1(\dot{r}_i),\ 1\}
$$

and $\ell_{iz},\ \tilde{\sigma}_{i\zeta},\ \tilde{\sigma}_{ir}$ be positive constants such that

$$\ell_{iz} \geq \max\{3\hat{c}_{ig}||P_{i\eta}G_i||^2,\ 1\},\quad \tilde{\sigma}_{i\zeta} \geq \max\{\ell_{iz}\sigma^1_{i\zeta},\ 3||P_{i\eta}F_iG_i||^2\},\quad \tilde{\sigma}_{ir} \geq \max\{\ell_{iz}\sigma^1_{ir},\ 3||P_{i\eta}k_{i1}||^2\}.$$

It follows then

$$\dot{V}_{i\tilde{z}} \leq -||\tilde{z}_i||^2 + \tilde{\sigma}_{i\zeta}\tilde{\gamma}_{i\zeta}(\zeta_i)||\zeta_i||^2 + \tilde{\sigma}_{ir}\tilde{\gamma}_{ir}(\dot{r}_i)||\dot{r}_i||^2.$$

Step 2: We consider the stability of the col$(\tilde{z}_i,\ \zeta_i)$-subsystem. Using the changing supply functions technique to this subsystem, one has that, for any given smooth $\Delta_{i\tilde{z}}(\tilde{z}_i) > 0$, there exists a continuously differentiable function $V^1_{i\tilde{z}}(\tilde{z}_i)$ satisfying that

$$\begin{aligned}&\underline{\alpha}_{i\tilde{z}}(||\tilde{z}_i||) \leq V^1_{i\tilde{z}}(\tilde{z}_i) \leq \overline{\alpha}_{i\tilde{z}}(||\tilde{z}_i||),\\ &\dot{V}^1_{i\tilde{z}} \leq -\Delta_{i\tilde{z}}(\tilde{z}_i)||\tilde{z}_i||^2 + \tilde{\sigma}^1_{i\zeta}\tilde{\gamma}^1_{i\zeta}(\zeta_i)||\zeta_i||^2 + \tilde{\sigma}^1_{ir}\tilde{\gamma}^1_{ir}(\dot{r}_i)||\dot{r}_i||^2\end{aligned}$$

for some known smooth functions $\underline{\alpha}_{i\tilde{z}},\ \overline{\alpha}_{i\tilde{z}} \in \mathcal{K}_\infty, \tilde{\gamma}^1_{i\zeta},\ \tilde{\gamma}^1_{ir} \geq 1$ and unknown constants $\tilde{\sigma}^1_{i\zeta},\ \tilde{\sigma}^1_{ir} \geq 1$.

Let $V_i(\tilde{z}_i,\ \zeta_i, \overline{W}_i,\ \zeta_i) = V^1_{i\tilde{z}}(\tilde{z}_i) + ||\zeta_i||^2 + \mathrm{Tr}(\overline{W}_i^\intercal\overline{W}_i) + \overline{\theta}_i^2$. It is positive definite and radially unbounded. Taking its derivative along the trajectory of (12) gives

$$\begin{aligned}\dot{V}_i \leq\ & -\Delta_{i\tilde{z}}(\tilde{z}_i)||\tilde{z}_i||^2 + \tilde{\sigma}^1_{i\zeta}\tilde{\gamma}^1_{i\zeta}(\zeta_i)||\zeta_i||^2 + \tilde{\sigma}^1_{ir}\tilde{\gamma}^1_{ir}(\dot{r}_i)||\dot{r}_i||^2 + 2\zeta_i^\intercal[\tilde{g}_i(z_i,\ \overline{\eta}_i,\ \zeta_i,\ r_i) - \overline{W}_i^\intercal\boldsymbol{\sigma}_i(r_i) - \theta_i^*\rho_i(\zeta_i)\zeta_i - \overline{\theta}_i\rho_i(\zeta_i)\zeta_i]\\ &+ 2\zeta_i^\intercal[\boldsymbol{\epsilon}_i(r_i) - k_{i1}\dot{r}_i] + 2\mathrm{Tr}(\overline{W}_i^\intercal[-\ell\overline{W}_i + \ell(W_i^* - W_i^0) + \boldsymbol{\sigma}_i(r_i)\zeta_i^\intercal]) + 2\overline{\theta}_i[-\ell(\overline{\theta}_i + \theta_i^* - \theta_i^0) + \rho_i(\zeta_i)||\zeta_i||^2]\\ \leq\ & -\Delta_{i\tilde{z}}(\tilde{z}_i)||\tilde{z}_i||^2 - 2\theta_i^*\rho_i(\zeta_i)||\zeta_i||^2 - 2\ell\mathrm{Tr}(\overline{W}_i^\intercal\overline{W}_i) - 2\ell||\overline{\theta}_i||^2 + \tilde{\sigma}^1_{i\zeta}\tilde{\gamma}^1_{i\zeta}(\zeta_i)||\zeta_i||^2 + 2\zeta_i^\intercal\tilde{g}_i(z_i,\ \overline{\eta}_i,\ \zeta_i,\ r_i)\\ &- 2\zeta_i^\intercal k_{i1}\dot{r}_i - 2\zeta_i^\intercal\boldsymbol{\epsilon}_i(r_i) - 2\ell\mathrm{Tr}(\overline{W}_i^\intercal(W_i^* - W_i^0)) - 2\ell\overline{\theta}_i^\intercal(\theta_i^* - \theta_i^0) + \tilde{\sigma}^1_{ir}\tilde{\gamma}^1_{ir}(\dot{r}_i)||\dot{r}_i||^2,\end{aligned}$$

where we use the identity $\mathrm{Tr}(ab^\intercal) = b^\intercal a$ for any two column vectors $a,\ b \in \mathbb{R}^n$.

Combining this inequality with (13), we further use Young's inequality and obtain that

$$\begin{aligned}\dot{V}_i \leq\ & -\Delta_{i\tilde{z}}(\tilde{z}_i)||\tilde{z}_i||^2 - 2\theta_i^*\rho_i(\zeta_i)||\zeta_i||^2 - 2\ell\mathrm{Tr}(\overline{W}_i^\intercal\overline{W}_i) - 2\ell||\overline{\theta}_i||^2 + \tilde{\sigma}^1_{i\zeta}\tilde{\gamma}^1_{i\zeta}(\zeta_i)||\zeta_i||^2 + \tilde{c}_{ig}||\zeta_i||^2\\ &+ [\tilde{\phi}_{i1}(\tilde{z}_i)||\tilde{z}_i||^2 + \tilde{\phi}_{i2}(\zeta_i)||\zeta_i||^2)] + ||\zeta_i||^2 + k_{i1}^2||\dot{r}_i||^2 + ||\zeta_i||^2 + ||\boldsymbol{\epsilon}_i(r_i)||^2\\ &- \ell\mathrm{Tr}(\overline{W}_i^\intercal\overline{W}_i) - \ell\mathrm{Tr}((W_i^* - W_i^0)^\intercal(W_i^* - W_i^0)) + \ell||\overline{\theta}_i||^2 + \ell||\theta_i^* - \theta_i^0||^2\\ \leq\ & -[\Delta_{i\tilde{z}}(\tilde{z}_i) - \tilde{\phi}_{i1}(\tilde{z}_i)]||\tilde{z}_i||^2 - \ell\mathrm{Tr}(\overline{W}_i^\intercal\overline{W}_i) - [2\theta_i^*\rho_i(\zeta_i) - \tilde{\sigma}^1_{i\zeta}\gamma^1_{i\zeta}(\zeta_i) - \tilde{\phi}_{i2}(\zeta_i) - \tilde{c}_{ig} - 2]||\zeta_i||^2\\ &- \ell||\overline{\theta}_i||^2 + c_{ir}||\dot{r}_i||^2 + ||\boldsymbol{\epsilon}_i(r_i)||^2 + \ell||W_i^* - W_i^0||_{\mathrm{F}}^2 + \ell||\theta_i^* - \theta_i^0||^2\end{aligned}$$

with $c_{ir} \triangleq \sup_{0\leq t\leq+\infty}|\tilde{\sigma}^1_{ir}\tilde{\gamma}^1_{ir}(\dot{r}_i(t))| + k_{i1}^2$. Note that this term c_{ir} is well-defined due to the boundedness of r_i and $\dot{r}_i$.

Select two smooth functions $\Delta_{i\tilde{z}},\ \rho_i$ such that

$$\Delta_{i\tilde{z}}(\tilde{z}_i) \geq 2\max\{\tilde{\phi}_{i1}(\tilde{z}_i),\ 1\},\quad \rho_i(\zeta_i) \geq \max\{\gamma^1_{i\zeta}(\zeta_i),\ \tilde{\phi}_{i2}(\zeta_i),\ 1\}$$

and a constant θ_i^* such that $\theta_i^* \geq \max\{\tilde{\sigma}^1_{i\zeta},\ \tilde{c}_{ig},\ 1\}$. It follows then

$$\dot{V}_i \leq -||\tilde{z}_i||^2 - ||\zeta_i||^2 - \ell\mathrm{Tr}(\overline{W}_i^\intercal\overline{W}_i) - \ell||\overline{\theta}_i||^2 + \Xi_i, \tag{14}$$

where $\Xi_i \triangleq c_{ir}||\dot{r}_i||^2 + \epsilon^2 + \ell||W_i^* - W_i^0||_{\mathrm{F}}^2 + \ell||\theta_i^* - \theta_i^0||^2$.

From the inequality (14), we can obtain that the first five subsystems in (12) is input-to-state stable with Ξ_i as its input by Theorem 4.19 in Khalil.[46] Since Ξ_i is upper bounded according to Theorem 2 in Liu and Wang,[35] we conclude the uniformly ultimate boundedness of trajectories of $\overline{\eta}_i$ and ξ_i according to Definition 4.7 in Khalil.[46] From the definitions of $\overline{\eta}_i$, ξ_i, and y^*, one can obtain the boundedness of state $[x]_i$ and coordination error $x_i(t) - y^*$. This completes the proof. ∎

Remark 1. This optimal coordination problem has been partially discussed in literature[9-11,20] for linear agents. By contrast, the high-order agents here are uncertain nonlinear ones and further subject to heterogeneous set constraints and external disturbances. Meanwhile, different from existing adaptive rules for uncertain multi-agent systems,[13,14] the proposed design methodology is a combination of neural network and internal model techniques. This joint design not only enables the controller to reject a large class of external disturbance but also facilitates us to successfully remove the restrictive linearly parameterized condition on nonlinearities required in existing results.[13]

Remark 2. From the expression of Ξ_i in inequality (14), both Ξ_i and the residual error $||x_i(t) - y^*||$ can be made smaller than any given positive constant by selecting a small enough l and increasing the number n_ω of neurons in the neural network. In this sense, this constrained optimal coordination problem for nonlinear multi-agent system (1) is solved by our distributed controller (10) in a globally practical sense.

4 | NUMERICAL EXAMPLE

In this section, we present two numerical examples to illustrate the effectiveness of our designs.

Example 1. Consider a group of single-link manipulators with flexible joints[46] modeled by:

$$\begin{aligned} J_{i1}\ddot{q}_{i1} + M_i g L_i \sin q_{i1} + k_i(q_{i1} - q_{i2}) &= 0, \\ J_{i2}\ddot{q}_{i2} - k_i(q_{i1} - q_{i2}) &= u_i + d_i, \quad i = 1, \dots, 4, \end{aligned} \tag{15}$$

where q_{i1}, q_{i2} are the angular positions, J_{i1}, J_{i2} are the moments of inertia, M_i is the total mass, L_i is a distance, k_i is a spring constant, u_i is the torque input, and d_i is the actuated disturbance of manipulator i. Suppose the information-sharing graph is depicted in Figure 2 with unity edge weights. It apparently satisfies Assumption 2. Letting $x_i = q_{i1}$, we can rewrite system (15) into the form (1) with $n_i = 4$, $b_i = k_i/(J_{i1}J_{i2})$, and

$$g_i([x]_i,\ L_i) = -x_i^{(2)}\left(\frac{M_i g L_i}{J_{i1}}\cos x_i + \frac{k_i}{J_{i1}} + \frac{k_i}{J_{i2}}\right) + \frac{M_i g L_i}{J_{i1}}\left(\dot{x}_i^2 - \frac{k_i}{J_{i2}}\right)\sin x_i.$$

We want to steer these manipulators to rendezvous at a common position that minimizes the aggregate distance from their starting position to this final position to save resources. For this purpose, we take the cost functions as $f_i(y) = \frac{1}{2}||y - q_{i1}(0)||^2$ and $f(y) = \frac{1}{2}\sum_{i=1}^4 ||y - q_{i1}(0)||^2$ $(i = 1, \dots, 4)$. To make this problem more interesting, we assume that $L_i = (1 + \mu_{i4})L_{i0}$ with nominal length L_{i0} and the external disturbances are described by

$$\begin{aligned} &D_1 = 1 + \mu_{15}, \quad S_1 = 0, \quad D_2 = 1 + \mu_{25}, \quad S_2 = 1, \\ &D_3 = [1 + \mu_{35}\ \ 0], \quad S_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad D_4 = [1\ \ 1 + \mu_{45}\ \ 0], \quad S_4 = \operatorname{diag}\left\{1, \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}\right\}, \end{aligned}$$

with unknown parameter μ_{ij}. Here $d_2(t)$ and $d_4(t)$ might tend to infinity depending the initial condition.

Note that feedback linearization rule fails to solve our problem due to the unknown parameters. Nevertheless, we can verify all assumptions in this article and thus develop a neural-network-based control (10) for this multi-agent system to

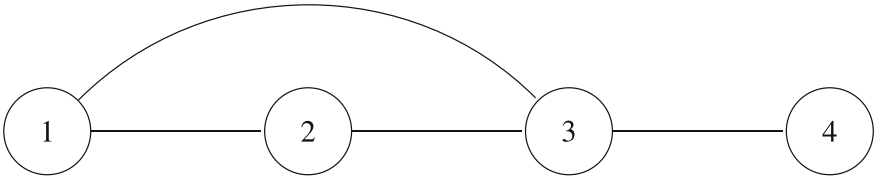

FIGURE 2 Communication graph $\mathcal{G}$ in our examples

solve this problem according to Theorem 1. To reject those external disturbances for agents, we choose

$$F_1 = -1, \quad G_1 = -1, \quad F_2 = \begin{bmatrix} -4 & 1 \\ -4 & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} -4 \\ -4 \end{bmatrix},$$
$$F_3 = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}, \quad G_3 = \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \quad F_4 = \begin{bmatrix} -4 & 2 \\ -2 & 0 \end{bmatrix}, \quad G_4 = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

for the internal model (8). In the simulations, we set $J_{i1} = J_{i2} = 1$, $L_{i0} = 1$, $M_i = 1$, $k_i = 1$ and assume that the uncertain parameter μ_{ij} is randomly chosen between −0.5 and 0.5. To approximate the unknown feedforwarding input, we construct the RBF neural network with the parameters $n_\omega = 21$, $\mu_{ij}^c = 0.5 * (j - 11)$, and $\kappa_i = 1.62$. The nonlinear control gain function is chosen as $\rho_i(s) = s^4 + 1$ with parameters $k_{i1} = 1$, $l = 0.01$ for $1 \le i \le 4$. All initial conditions are randomly chosen. The simulation result is shown in Figure 3, where $x_i(t)$ is found to quickly converge to the neighborhood of the optimal point $y^\star = \frac{1}{4}\sum_{i=1}^{4} q_{i1}(0)$ with small residual errors.

Example 2. Consider another multi-agent system including two controlled Van der Pol oscillators

$$\begin{aligned} \dot{x}_{i1} &= x_{i2}, \\ \dot{x}_{i2} &= -(1 + \mu_{i1})x_{i1} + (1 + \mu_{i2})(\mu_{i3} - x_{i1}^2)x_{i2} + u_i + d_i, \quad i = 1,\ 2 \end{aligned}$$

and two controlled Duffing equations

$$\begin{aligned} \dot{x}_{i1} &= x_{i2}, \\ \dot{x}_{i2} &= -(1 + \mu_{i1})x_{i1}(\mu_{i2} - x_{i1}^2) - (1 + \mu_{i3})x_{i2} + u_i + d_i, \quad i = 3,\ 4 \end{aligned}$$

with input u_i, output x_{i1}, and disturbance d_i. Assume the disturbances are generated by (2) with

$$D_i = 1 + \mu_{i4}, \quad S_i = \begin{bmatrix} 0 & 1 \\ -i & 0 \end{bmatrix}, \quad i = 1, \ldots, 4$$

and the unknown parameter μ_{ij} is randomly chosen between −0.5 and 0.5. The information-sharing graph is taken as the same as Example 1.

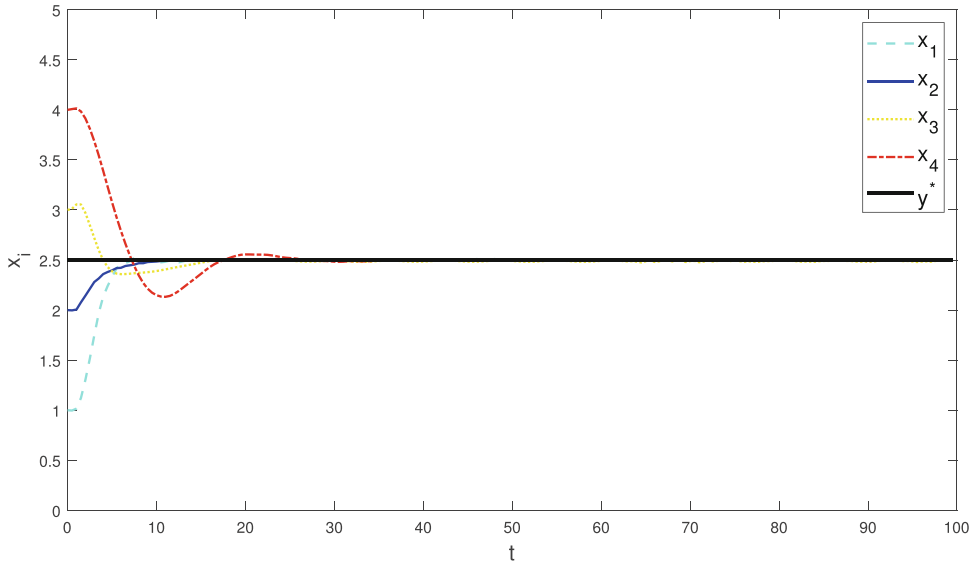

FIGURE 3 Profiles of agent outputs in Example 1

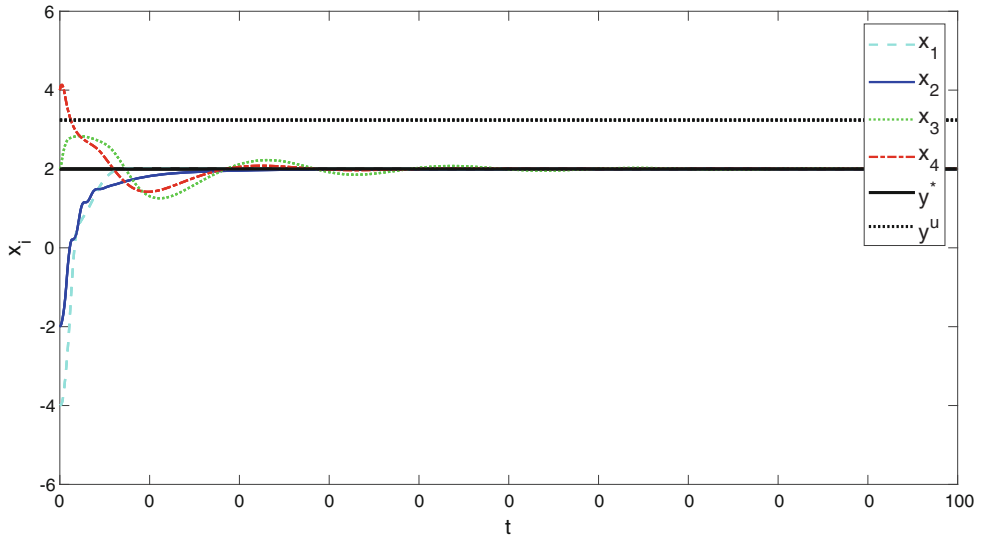

FIGURE 4 Profiles of agent outputs in Example 2

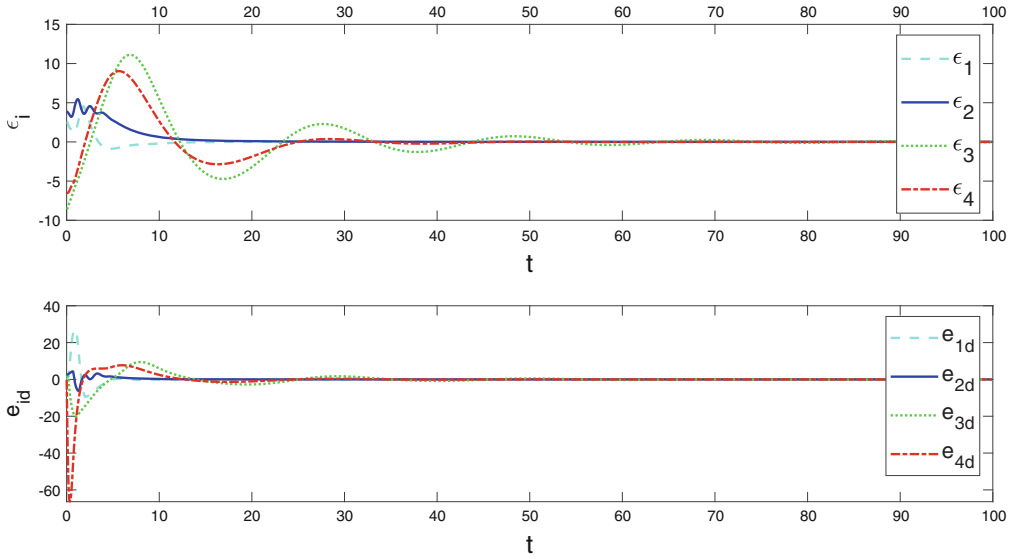

FIGURE 5 Profiles of ϵ_i and $e_{id} = u_{id}(t) + d_i(t)$ in Example 2

Although all agents are of the form (1) with $x_i = x_{i1}$, $n_i = 2$, and $q = 1$, the two classes of agent dynamics possess very different behaviors. This heterogeneity definitely brings many challenges in resolving their coordination problem. Moreover, we choose some complex cost functions as $f_1(y) = (y-8)^2$, $f_2(y) = y^2/(80\log(y^2+2)) + (y-5)^2$, $f_3(y) = y^2/(20\sqrt{y^2+1}) + y^2$, $f_4(y) = \log\left(e^{-0.05y} + e^{0.05y}\right) + y^2$ with local interval constraints $[-3+i,\ 1+i\,]$. Assumption 1 is fulfilled with $\underline{l}_i = 1$ and $\bar{l}_i = 3$ for $i = 1, \ldots, 4$. Then, the formulated coordination problem for these agents can be solved by a controller of the form (10). In fact, the optimal solution to the global constrained optimization problem is $y^* = 2$ while the unconstrained optimal point is $y^u = 3.24$ by directly minimizing $\sum_{i=1}^4 f_i(y)$.

For simulations, we choose the following matrices

$$F_i = \begin{bmatrix} -2i & 1 \\ -i & 0 \end{bmatrix}, \quad G_i = \begin{bmatrix} -2i \\ 0 \end{bmatrix}, \quad i = 1, \ldots, 4$$

for the internal model (8) and use the same RBF neural network as in Example 1. The control gain function is chosen as $\rho_i(s) = s^6 + 1$ with parameters $k_{i1} = 1$, $l = 0.001$ for $1 \leq i \leq 4$. With randomly chosen initial conditions, the profiles of agent outputs under controller (10) are shown in Figure 4. We also list the approximation errors of feedforwarding input and external disturbance by neural networks and internal models in Figure 5. It can be found that both $\varepsilon_i(t)$ and $e_{id}(t)$ converge toward zero as t grows. The performance verifies the effectiveness of controller (10) to ensure the expected constrained optimal coordination for this heterogeneous uncertain multi-agent system (1).

5 | CONCLUSION

We have investigated the constrained optimal coordination problem for a class of heterogeneous nonlinear agents subject to both unknown dynamics and external disturbances. Jointly with internal-model-based designs, novel distributed neural-network-based controllers have been developed to overcome the technical difficulties brought by uncertainties, disturbances, and decision constraints under some standard assumptions. Output feedback control for more general multi-agent systems and communication graphs will be our future work.

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CONFLICT OF INTEREST

We confirm that there are no conflict of interests for this article.

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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