

Optimal Output Consensus for Nonlinear Multi-agent Systems with Both Static and Dynamic Uncertainties

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Abstract—In this technical note, we investigate an optimal output consensus problem for heterogeneous uncertain nonlinear multi-agent systems. The considered agents are described by high-order nonlinear dynamics subject to both static and dynamic uncertainties. A two-step design, comprising sequential constructions of optimal signal generator and distributed partial stabilization feedback controller, is developed to overcome the difficulties brought by nonlinearities, uncertainties, and optimal requirements. Our study can not only assure an output consensus, but also achieve an optimal agreement characterized by a distributed optimization problem.

Index Terms—Optimal output consensus, multi-agent system, distributed optimization, uncertainties, adaptive control

I. INTRODUCTION

In the past few years, distributed optimization has attracted much attention due to its broad potential applications in multi-robot systems, smart grid and sensor networks. In a typical setting, each agent has access to a private objective function and all agents are regulated to achieve a consensus on the optimal solution of the sum of all local functions. Many important results were obtained based on gradients or subgradients of the local objective functions combined with consensus rules, including both discrete-time and continuous-time algorithms [1]–[6].

Since distributed optimization tasks may be implemented or depend on physical dynamics in practice, optimal consensus involving high-order agent dynamics deserves further investigation. Compared with the pure (output) consensus problem, the consensus point for all outputs of agents is additionally required to be an optimal solution of the global cost function. Note that this optimal solution can only be determined and reached in a distributed way. Some interesting attempts have been made in [7]–[9] for integrator agents, [10] for linear agents, and [11], [12] for special classes of nonlinear agents. However, optimal output consensus for more general nonlinear multi-agent systems is still far from being solved, especially for agents being heterogeneous and subject to uncertainties.

In this paper, we consider nonlinear multi-agent systems in the Byrnes-Isidori normal form which can model many typical

mechanical and electromechanical systems [13]. In literature, there have been many consensus results for agents of this type, e.g., [14]–[16]. This normal form is general enough to cover the dynamics reported in existing optimal consensus results [7]–[12], [17], [18]. Here, we further take into account heterogeneous nonlinear dynamics having both static and dynamic uncertainties, which inevitably bring technical difficulties in resolving the optimal output consensus problem. In a preliminary work [19], this problem was studied for such class of agents assuming that the compact set containing static uncertainties is prior known. In this present study, we remove such restrictive condition and allow the boundary of this compact set to be unknown.

The contribution of this paper is at least two-fold. First, we solve the optimal output consensus problem for a larger class of uncertain nonlinear multi-agent systems, significantly improving the existing results reported in [9]–[12]. Second, a novel dynamic compensator based distributed controller is developed for effectively addressing complicated uncertainties, while precise information of system dynamics is required in [7]–[9]. Moreover, in contrast with relevant results in [11], [19], the boundary of the compact set containing uncertain parameters is allowed to be unknown.

The rest of this paper is organized as follows. Preliminaries and problem formulation are presented in Section II. Then, the design scheme and main results are provided in Sections III and IV with an illustrative example in Section V. Finally, conclusions are given in Section VI.

Notation: Let \mathbb{R}^N be the N -dimensional Euclidean space. Denote $\text{col}(a_1, \dots, a_N) = [a_1^T, \dots, a_N^T]^T$ for vectors a_1, \dots, a_N . $\mathbf{1}_N$ (or $\mathbf{0}_N$) denotes an all-one (or all-zero) vector in \mathbb{R}^N and I_N denotes the $N \times N$ identity matrix. Let $M_1 = \frac{1}{\sqrt{N}}\mathbf{1}_N$ and M_2 be the matrix satisfying $M_2^T M_1 = \mathbf{0}_{N-1}$, $M_2^T M_2 = I_{N-1}$, and $M_2 M_2^T = I_N - M_1 M_1^T$. Denote the Euclidean norm of vector a by $\|a\|$ and the spectral norm of matrix A by $\|A\|$. A continuous function $\alpha: [0, +\infty) \rightarrow [0, +\infty)$ belongs to class \mathcal{K} if it is strictly increasing and $\alpha(0) = 0$; It further belongs to class \mathcal{K}_∞ if it belongs to class \mathcal{K} and $\lim_{s \rightarrow \infty} \alpha(s) = \infty$.

II. PRELIMINARIES AND PROBLEM FORMULATION

In this section, we present preliminaries of partial stability and graph theory, and then the formulation of our problem.

A. Partial stability

To achieve optimal output consensus, we need to ensure the convergence of particular partial state of the closed-loop

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system rather than the full state. Such an issue is often referred to as partial stability (stabilization) [20]. Since the closed-loop system may have a continuum of equilibria, we introduce a modified version of partial stability as follows.

Consider the nonlinear autonomous system

$$\dot{x}_1 = f_1(x_1, x_2), \quad \dot{x}_2 = f_2(x_1, x_2) \quad (1)$$

where $x = \text{col}(x_1, x_2)$ with $x_1 \in \mathbb{R}^{n_{x_1}}$, $x_2 \in \mathbb{R}^{n_{x_2}}$ and the functions f_1, f_2 are sufficiently smooth. Denote the equilibria set as $\mathcal{D} \triangleq \{x \mid f_1(x_1, x_2) = \mathbf{0}, f_2(x_1, x_2) = \mathbf{0}\}$.

Definition 1: System (1) is Lyapunov semistable with respect to x_1 (or briefly, x_1 -semistable) at x_1^* if, for every $\varepsilon > 0$, there exist x_2^* and $\delta > 0$ such that $x^* = \text{col}(x_1^*, x_2^*) \in \mathcal{D}$ and $\|x(0) - x^*\| < \delta$ implies $\|x_1(t) - x_1^*\| \leq \varepsilon$ for all $t \geq 0$. If for any $x(0)$, it further holds that $\lim_{t \rightarrow +\infty} \|x_1(t) - x_1^*\| = 0$, this system is globally asymptotically x_1 -semistable at x_1^* .

When $\mathcal{D} = \{\mathbf{0}\}$, this definition is exactly the partial stability concept with respect to x_1 specified in [22, page 17]. The following lemma is slightly modified from Theorems 4.5 and 4.7 in [21] and its proof is omitted.

Lemma 1: Suppose that there exist a continuously differentiable function $V(x)$ and a constant vector $x_2^* \in \mathbb{R}^{n_{x_2}}$ such that $x^* = \text{col}(x_1^*, x_2^*) \in \mathcal{D}$, and along the trajectory of (1),

$$\begin{aligned} \alpha(\|x - x^*\|) &\leq V(x) \leq \beta(\|x - x^*\|) \\ \dot{V}(x) &\leq -\gamma(\|x_1 - x_1^*\|) \end{aligned}$$

for some functions $\alpha, \beta \in \mathcal{K}_\infty$ and $\gamma \in \mathcal{K}$. Then, system (1) admits well-defined bounded trajectories over $[0, +\infty)$ and is globally asymptotically x_1 -semistable at x_1^* .

B. Graph notion

A weighted directed graph (digraph) is described by $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$ with node set $\mathcal{N} = \{1, \dots, N\}$ and edge set \mathcal{E} . $(i, j) \in \mathcal{E}$ denotes an edge from node i to node j . The weighted adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ is defined by $a_{ii} = 0$ and $a_{ij} \geq 0$. Here $a_{ij} > 0$ iff there is an edge (j, i) in the digraph. Node i 's neighbor set is defined as $\mathcal{N}_i = \{j \mid (j, i) \in \mathcal{E}\}$. We denote $\mathcal{N}_i^0 = \mathcal{N}_i \cup \{i\}$. A directed path is an alternating sequence $i_1 e_1 i_2 e_2 \dots e_{k-1} i_k$ of nodes i_l and edges $e_m = (i_m, i_{m+1}) \in \mathcal{E}$ for $l = 1, 2, \dots, k$. If there is a directed path between any two nodes, then the digraph is said to be strongly connected. The in-degree and out-degree of node i are defined by $d_i^{\text{in}} = \sum_{j=1}^N a_{ij}$ and $d_i^{\text{out}} = \sum_{j=1}^N a_{ji}$. A digraph is weight-balanced if $d_i^{\text{in}} = d_i^{\text{out}}$ for any $i \in \mathcal{N}$. The Laplacian of \mathcal{G} is defined as $L \triangleq D^{\text{in}} - \mathcal{A}$ with $D^{\text{in}} = \text{diag}(d_1^{\text{in}}, \dots, d_N^{\text{in}})$. Note that $L\mathbf{1}_N = \mathbf{0}_N$ for any digraph. If this digraph is weight-balanced, we have $\mathbf{1}_N^T L = \mathbf{0}_N^T$ and matrix $\text{Sym}(L) \triangleq \frac{L+L^T}{2}$ is positive semidefinite. For a weight-balanced and strongly connected digraph, we can order the eigenvalues of $\text{Sym}(L)$ as $0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_N$ and have $\lambda_2 I_{N-1} \leq M_2^T \text{Sym}(L) M_2 \leq \lambda_N I_{N-1}$. See [22] for more details.

C. Problem formulation

Consider a group of nonlinear systems modeled by

$$\begin{aligned} \dot{z}_i &= h_i(z_i, y_i, w) \\ \dot{x}_i &= A_i x_i + B_i [g_i(z_i, x_i, w) + b_i(w) u_i] \\ y_i &= C_i x_i, \quad i = 1, \dots, N \end{aligned} \quad (2)$$

where $\text{col}(z_i, x_i)$ is the state with $x_i = \text{col}(x_{i1}, \dots, x_{in_i}) \in \mathbb{R}^{n_i}$ and $z_i \in \mathbb{R}^{m_i}$, $u_i \in \mathbb{R}$ is the input, $y_i \in \mathbb{R}$ is the output, and $w \in \mathbb{W} \subset \mathbb{R}^{n_w}$ with \mathbb{W} being compact and containing the origin. The triplet (C_i, A_i, B_i) represents a chain of n_i integrators in canonical form, that is,

$$A_i = \begin{bmatrix} \mathbf{0}_{n_i-1} & I_{n_i-1} \\ 0 & \mathbf{0}_{n_i-1}^T \end{bmatrix}, B_i = \begin{bmatrix} \mathbf{0}_{n_i-1} \\ 1 \end{bmatrix}, C_i = \begin{bmatrix} 1 \\ \mathbf{0}_{n_i-1} \end{bmatrix}^T$$

Here w and z_i represent static and dynamic uncertainties of agent i , respectively. Different from [19], the compact set \mathbb{W} containing the static uncertainties is not necessarily known here. It is assumed that all functions are sufficiently smooth and satisfy $h_i(\mathbf{0}, \mathbf{0}, w) = 0$, $g_i(\mathbf{0}, \mathbf{0}, w) = 0$, $b_i(w) \geq b_0 > 0$ for all $w \in \mathbb{W}$ with some constant b_0 .

We endow each agent output with a local cost function $f_i: \mathbb{R} \rightarrow \mathbb{R}$, and define the global cost function as the sum of all local costs, i.e., $f(y) = \sum_{i=1}^N f_i(y)$. For multi-agent system (2), we aim to develop an algorithm such that all agent outputs achieve a consensus on the minimizer to this global cost function in a distributed fashion. For this purpose, a digraph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$ is used to describe the information communication among agents with node set $\mathcal{N} = \{1, \dots, N\}$, edge set $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$, and weighted matrix $\mathcal{A} \in \mathbb{R}^{N \times N}$. An edge $(j, i) \in \mathcal{E}$ with weight $a_{ij} > 0$ means that agent i can get the information of agent j .

The considered distributed controller is described by

$$\begin{aligned} u_i &= \Xi_{i1}(\nabla f_i, x_j, \chi_j, j \in \mathcal{N}_i^0) \\ \dot{\chi}_i &= \Xi_{i2}(\nabla f_i, x_j, \chi_j, j \in \mathcal{N}_i^0) \end{aligned} \quad (3)$$

where $\chi_i \in \mathbb{R}^{q_i}$ is the compensator state and Ξ_{i1}, Ξ_{i2} are smooth functions to be specified later. With these preparations, we formulate our problem explicitly as follows.

Problem 1: For multi-agent system (2), function f_i , digraph \mathcal{G} , and compact set \mathbb{W} , find a controller of the form (3) such that, for each $w \in \mathbb{W}$ and each initial condition $\text{col}(z_i(0), x_i(0), \chi_i(0)) \in \mathbb{R}^{m_i+n_i+q_i}$,

- the trajectory of the closed-loop system composed of (2) and (3) exists and is bounded over $[0, +\infty)$;
- the outputs of agents satisfy $\lim_{t \rightarrow +\infty} |y_i(t) - y^*| = 0$ with y^* being optimal solution of

$$\min_{y \in \mathbb{R}} f(y) = \sum_{i=1}^N f_i(y) \quad (4)$$

Remark 1: Compared to existing output consensus results [14]–[16], this problem further requires the outputs of agents to reach an agreement on the optimal point y^* specified by minimizing a cost function. In this sense, we say these agents achieving an optimal output consensus as in [2], [8], [9].

This problem for single integrators has been coined as distributed optimization and investigated for many years. For high-order nonlinear agents, it is certainly more challenging to achieve such an optimal output consensus, while the static and dynamic uncertainties bring extra technical difficulties in resolving this problem.

III. TWO-STEP DESIGN SCHEME

In this section, we convert the optimal output consensus problem into a distributed partial stabilization problem by

constructing optimal signal generators, giving rise to a two-step design scheme for solving Problem 1.

To begin with, several standing assumptions are listed.

Assumption 1: The digraph \mathcal{G} is weight-balanced and strongly connected.

Assumption 2: For each $i \in \mathcal{N}$, the function f_i is twice continuously differentiable and satisfies that $L_i \leq \nabla^2 f_i(s) \leq \bar{L}_i$ with constants $0 < \underline{L}_i \leq \bar{L}_i < +\infty$ for all $s \in \mathbb{R}$.

Assumption 3: For each $i \in \mathcal{N}$, there exists a smooth function $z_i^*(s, w)$ satisfying $z_i^*(0, w) = 0$ and $h_i(z_i^*(s, w), s, w) = 0$ for all $s \in \mathbb{R}$ and $w \in \mathbb{R}^{n_w}$.

Assumption 1 guarantees that each agent's information can be reached by any other agent. Assumption 2 implies the existence and uniqueness of optimal solution to problem (4) [23]. Assumption 3 can be interpreted as the solvability of regulator equations in the context of output regulation [24]. These assumptions have been widely used in (distributed) coordination for multi-agent systems [3], [7], [11], [25], [26].

Consider an optimal consensus problem for a group of single integrators with the same optimal requirement (4)

$$\dot{r}_i = \mu_i \quad (5)$$

If this auxiliary problem is solved by some chosen μ_i , we only need to drive agent i to track the generated signal $r_i(t)$ to achieve the optimal output consensus for agent (2).

Since the Laplacian L of digraph \mathcal{G} is asymmetric, the generator in [10] fails to reproduce y^* without the information of L^T . Motivated by [18], we present a candidate of optimal signal generator for problem (4) as follows

$$\begin{aligned} \mu_i &= -\alpha \nabla f_i(r_i) - \beta \sum_{j=1}^N a_{ij}(r_i - r_j) - \sum_{j=1}^N a_{ij}(v_i - v_j) \\ \dot{v}_i &= \alpha \beta \sum_{j=1}^N a_{ij}(r_i - r_j) \end{aligned} \quad (6)$$

where α, β are constants to be specified later. Putting it into a compact form gives

$$\dot{r} = -\alpha \nabla \tilde{f}(r) - \beta Lr - Lv, \quad \dot{v} = \alpha \beta Lr \quad (7)$$

where $r = \text{col}(r_1, \dots, r_N)$, $v = \text{col}(v_1, \dots, v_N)$, and function $\tilde{f}(r) \triangleq \sum_{i=1}^N f_i(r_i)$ is L -strongly convex while its gradient $\nabla \tilde{f}(r)$ is \bar{L} -Lipschitz with $\bar{L} = \max_i \{\bar{L}_i\}$ and $\underline{L} = \min_i \{\underline{L}_i\}$.

Let $\text{col}(r^*, v^*)$ be the equilibrium point of system (7). It is verified that $r^* = \mathbf{1}_N y^*$ under Assumptions 1 and 2 by Theorem 3.27 in [23]. For (7), we have the following interesting result.

Lemma 2: Suppose Assumptions 1–2 hold and let $\alpha \geq \max\{1, \frac{1}{\underline{L}}, \frac{2\bar{L}^2}{\underline{L}\lambda_2}\}$, $\beta \geq \max\{1, \frac{1}{\lambda_2}, \frac{6\alpha^2 \lambda_N^2}{\lambda_2^2}\}$. Then, system (7) admits well-defined bounded trajectories over $[0, +\infty)$ and is globally asymptotically r -semistable at $\mathbf{1}_N y^*$. Moreover, $r_i(t)$ approaches y^* exponentially as $t \rightarrow +\infty$ for $i \in \mathcal{N}$.

Proof: Briefly, we utilize Lemma 1 to complete the proof. Let $M_L = M_2^T L M_2$ and $v^* = -\alpha M_2 M_L^{-1} M_2^T \nabla \tilde{f}(r^*)$. It can be verified that $\text{col}(r^*, v^*)$ is an equilibrium of system (7).

Perform the coordinate transformation: $\bar{r}_1 = M_1^T (r - r^*)$, $\bar{r}_2 = M_2^T (r - r^*)$, $\bar{v}_1 = M_1^T (v - v^*)$, and $\bar{v}_2 = M_2^T [(v + \alpha r) - (v^* + \alpha r^*)]$. It follows that $\dot{\bar{v}}_1 = 0$ and

$$\begin{aligned} \dot{\bar{r}}_1 &= -\alpha M_1^T \Pi \\ \dot{\bar{r}}_2 &= -\alpha M_2^T \Pi - \beta M_L \bar{r}_2 + \alpha M_L \bar{r}_2 - M_L \bar{v}_2 \\ \dot{\bar{v}}_2 &= -\alpha M_L \bar{v}_2 + \alpha^2 M_L \bar{r}_2 - \alpha^2 M_2^T \Pi \end{aligned} \quad (8)$$

where $\Pi \triangleq \nabla \tilde{f}(r) - \nabla \tilde{f}(r^*)$. Let $\bar{r} = \text{col}(\bar{r}_1, \bar{r}_2)$, and $V_0(r, v) = \bar{r}^T \bar{r} + \frac{1}{\alpha^3} \bar{v}_1^T \bar{v}_1 + \frac{1}{\alpha^2} \bar{v}_2^T \bar{v}_2$ in this new coordinate with $\alpha > 0$ to be specified later. The first inequality in Lemma 1 apparently hold. On the other hand, by Young's inequality, the time derivative of V_0 along the trajectory of (7) satisfies

$$\begin{aligned} \dot{V}_0 &= -2\alpha(r - r^*)^T \Pi + 2\bar{r}_2^T [-\beta M_L \bar{r}_2 + \alpha M_L \bar{r}_2 - M_L \bar{v}_2] \\ &\quad + \frac{2}{\alpha^3} \bar{v}_1^T [-\alpha M_L \bar{v}_2 + \alpha^2 M_L \bar{r}_2 - \alpha^2 M_2^T \Pi] \\ &\leq -2\alpha \|\bar{r}\|^2 - 2\beta \lambda_2 \|\bar{r}_2\|^2 + 2\alpha \lambda_N \|\bar{r}_2\|^2 + 2\lambda_N \|\bar{r}_2\| \|\bar{v}_2\| \\ &\quad - \frac{2\lambda_2}{\alpha^2} \|\bar{v}_2\|^2 + \frac{2}{\alpha} \lambda_N \|\bar{r}_2\| \|\bar{v}_2\| + \frac{2\bar{L}}{\alpha} \|\bar{v}_2\| \|\bar{r}\| \\ &\leq -(2\alpha \underline{L} - \frac{3\bar{L}^2}{\lambda_2}) \|\bar{r}\|^2 - \frac{\lambda_2}{\alpha^2} \|\bar{v}_2\|^2 \\ &\quad - (2\beta \lambda_2 - 2\alpha \lambda_N - \frac{3\alpha^2 \lambda_N^2}{\lambda_2} - \frac{3\lambda_N^2}{\lambda_2}) \|\bar{r}_2\|^2 \\ &\leq -\frac{1}{2} \|\bar{r}\|^2 - \frac{1}{2\alpha^3} \|\bar{v}_2\|^2 \triangleq W_0(\bar{r}, \bar{v}_2) \end{aligned}$$

According to Lemma 1, we conclude the boundedness of all trajectories over $[0, +\infty)$ and its r -semistability of system (7) at $\mathbf{1}_N y^*$. By further considering the reduced-order system (8) with a Lyapunov function $W_0(\bar{r}, \bar{v}_2)$, one can obtain that $\dot{W}_0 \leq -\frac{1}{2} W_0$ along the trajectories of (8). Recalling Theorem 4.10 in [13], $W_0(\bar{r}(t), \bar{v}_2(t))$ and thus $\bar{r}(t)$ exponentially converge to 0 as t goes to infinity. The proof is complete. ■

Remark 2: The optimal signal generator (7) is a modified version of the augmented Lagrangian method solving problem (4) in [18]. Here we add an extra parameter α to simplify both the synthesis and its analysis. Compared with the results for digraphs in [8], [9], [17], our algorithm is initialization-free to generate the optimal point y^* . This makes it possible to work in a scalable manner, which might be favorable for dynamic networks with leaving-off and plugging-in of agents.

Remark 3: In our design, we use the knowledge of λ_2 and λ_N as that in [7], [25] to compensate the asymmetry of directed information flows. It should be mentioned that these values can be computed by existing algorithms beforehand, e.g., [27].

Under Assumption 3, we denote $x_i^*(r_i) = \text{col}(r_i, \mathbf{0}_{n_i-1})$, $u_i^*(r_i, w) = -\frac{g_i(z_i^*(r_i, w), x_i^*(r_i, w))}{b_i(w)}$ and perform the coordinate transformation: $\bar{z}_i = z_i - z_i^*(r_i, w)$, $\bar{x}_i = x_i - x_i^*(r_i)$. This leads to an interconnected error system as follows

$$\begin{aligned} \dot{\bar{z}}_i &= \bar{h}_i(\bar{z}_i, e_i, r_i, w) - \frac{\partial z_i^*}{\partial r_i} \mu_i \\ \dot{\bar{x}}_i &= A_i \bar{x}_i + B_i [\bar{g}_i(\bar{z}_i, \bar{x}_i, r_i, w) \\ &\quad + b_i(w)(u_i - u_i^*(r_i, w))] - E_i \mu_i \\ e_i &= C_i \bar{x}_i, \quad i \in \mathcal{N} \end{aligned} \quad (9)$$

where $E_i = \text{col}(1, \mathbf{0}_{n_i-1})$ and

$$\begin{aligned} \bar{h}_i(\bar{z}_i, e_i, r_i, w) &= h_i(z_i, y_i, w) - h_i(z_i^*(r_i, w), r_i, w) \\ \bar{g}_i(\bar{z}_i, \bar{x}_i, r_i, w) &= g_i(z_i, x_i, w) - g_i(z_i^*(r_i, w), x_i^*(r_i, w) \end{aligned}$$

It can be verified that $\bar{h}_i(\mathbf{0}, \mathbf{0}, r_i, w) = 0$, $\bar{g}_i(\mathbf{0}, \mathbf{0}, r_i, w) = 0$ for all $r_i \in \mathbb{R}$ and $w \in \mathbb{R}^{n_w}$.

Attaching the optimal signal generator (7) to error system (9) yields an augmented system associated with Problem 1.

A key lemma is obtained to assist us in solving the optimal output consensus problem.

Lemma 3: Suppose Assumptions 1–3 hold and there exists a smooth controller of the form

$$\begin{aligned} u_i &= \Xi_{i1}^0(\bar{x}_j, r_j, \chi_j^0, j \in \mathcal{N}_i^0) \\ \dot{\chi}_i^0 &= \Xi_{i2}^0(\bar{x}_j, r_j, \chi_j^0, j \in \mathcal{N}_i^0) \end{aligned} \quad (10)$$

solving the distributed partial stabilization problem of the augmented system composed of (7) and (9) in the sense that the closed-loop system composed of (7), (9), and (10) admits well-defined bounded trajectories over $[0, +\infty)$ and is globally asymptotically e_i -semistable at 0. Then, Problem 1 can be solved by a controller composed of (6) and (10).

Proof: Under the lemma condition, we can confirm that trajectories of all agents are well-defined bounded over $[0, +\infty)$ and $\lim_{t \rightarrow +\infty} e_i(t) = 0$ for any initial condition $\text{col}(\bar{z}_i(0), \bar{x}_i(0), \chi_i^0(0), r(0), v(0))$. Note that $|y_i(t) - y^*| \leq |e_i(t)| + |r_i(t) - y^*|$ by the triangle inequality. This together with Lemma 2 ensures that $\lim_{t \rightarrow +\infty} |y_i(t) - y^*| = 0$. ■

Remark 4: Based on Lemma 3, our optimal output consensus problem for multi-agent system (2) is converted into a distributed partial stabilization problem of certain interconnected augmented systems. As the considered nonlinear multi-agent system (2) is further subject to static and dynamic uncertainties, the associated partial stabilization design is more challenging than relevant results obtained in [9]–[12]. On the other hand, existing designs presented in [20], [21] are not applicable for such complicated uncertainties and the partial stabilization problem itself is nontrivial even for a single nonlinear system. Thus, we have to seek a robust distributed partial stabilization design method for the augmented systems.

IV. MAIN RESULT

In this section, we focus on the subsequent partial stabilization problem of the augmented system composed of (7) and (9) and eventually solve the optimal output consensus problem for multi-agent system (2).

To this end, we make an extra assumption imposing a mild minimum-phase condition widely used in nonlinear stabilization problems [26], [28], [29].

Assumption 4: For each $i \in \mathcal{N}$, there exists a continuously differentiable function $W_{\bar{z}}(\bar{z}_i)$ such that, for all $r_i \in \mathbb{R}$ and $w \in \mathbb{W}$, along the trajectory of system (9),

$$\begin{aligned} \underline{\alpha}_i(\|\bar{z}_i\|) &\leq W_{\bar{z}}(\bar{z}_i) \leq \bar{\alpha}_i(\|\bar{z}_i\|) \\ \dot{W}_{\bar{z}} &\leq -\alpha_i(\|\bar{z}_i\|) + \sigma_{ie} \gamma_{ie}(e_i) e_i^2 + \sigma_{i\mu} \gamma_{i\mu}(r_i) \mu_i^2 \end{aligned} \quad (11)$$

for some known smooth functions $\underline{\alpha}_i, \bar{\alpha}_i, \alpha_i \in \mathcal{K}_\infty, \gamma_{ie}, \gamma_{i\mu} > 1$, and unknown constants $\sigma_{ie}, \sigma_{i\mu} > 1$ with α_i satisfying $\limsup_{s \rightarrow 0^+} \frac{\alpha_i^{-1}(s^2)}{s} < +\infty$.

Due to the presence of uncertain parameter w , the feedforward term $u_i^*(r_i, w)$ is unavailable for feedback. To tackle this issue, we introduce a dynamic compensator as follows

$$\dot{\eta}_i = -\kappa_i(r_i) \eta_i + u_i$$

where $\kappa_i(r_i) > 0$ is a smooth function to be specified later. Here, $\kappa_i(r_i)$ is a scaling factor to handle nonlinear functions

of r_i . This compensator reduces to an internal model when $\kappa_i(r_i)$ is constant [24].

Consider the error system (9). For $n_i \geq 2$, choose constants k_{ij} such that the polynomial $p_i(\lambda) = \sum_{j=1}^{n_i-1} k_{ij} \lambda^{j-1} + \lambda^{n_i-1}$ is Hurwitz. Let $\xi_i = \text{col}(\bar{x}_{i1}, \dots, \bar{x}_{in_i-1})$, $\zeta_i = \sum_{j=1}^{n_i-1} k_{ij} \bar{x}_{ij} + \bar{x}_{in_i}$, and $\beta_i(\eta_i, r_i) \triangleq \kappa_i(r_i) \eta_i$. Performing coordinate and input transformations: $\bar{\eta}_i = \eta_i - \frac{u_i^*(r_i, w)}{\kappa_i(r_i)} - b_i^{-1}(w) \zeta_i$ and $\bar{u}_i = u_i - \beta_i(\eta_i, r_i)$ gives a composite system in the following form

$$\begin{aligned} \dot{\bar{z}}_i &= \bar{h}_i(\bar{z}_i, e_i, r_i, w) - \frac{\partial \bar{z}_i^*}{\partial r_i} \mu_i \\ \dot{\xi}_i &= A_i^0 \xi_i + B_i^0 \zeta_i - E_i^0 \mu_i \\ \dot{\bar{\eta}}_i &= -\kappa_i(r_i) \bar{\eta}_i + \bar{g}_i(\bar{z}_i, \xi_i, \zeta_i, r_i, w) + \psi_i(r_i, w) \mu_i \\ \dot{\zeta}_i &= \check{g}_i(\bar{z}_i, \xi_i, \bar{\eta}_i, \zeta_i, r_i, w) + b_i(w) \bar{u}_i - k_{i1} \mu_i \end{aligned} \quad (12)$$

where

$$\begin{aligned} A_i^0 &= \begin{bmatrix} \mathbf{0}_{n_i-2} & I_{n_i-2} \\ -k_{i1} & -k_{i2}, \dots, -k_{in_i-1} \end{bmatrix} \\ B_i^0 &= \begin{bmatrix} \mathbf{0}_{n_i-2} \\ 1 \end{bmatrix}, \quad E_i^0 = \begin{bmatrix} 1 \\ \mathbf{0}_{n_i-2} \end{bmatrix} \\ \bar{g}_i &= -\frac{1}{b_i(w)} [\hat{g}_i(\bar{z}_i, \xi_i, \zeta_i, r_i, w) + \kappa_i(r_i) \zeta_i] \\ \psi_i &= \frac{u_i^*(r_i, w)}{\kappa_i^2(r_i)} \frac{\partial \kappa_i(r_i)}{\partial r_i} - \frac{1}{\kappa_i(r_i)} \frac{\partial u_i^*(r_i, w)}{\partial r_i} + \frac{k_{i1}}{b_i(w)} \\ \check{g}_i &= \kappa_i(r_i) \zeta_i + b_i(w) \kappa_i(r_i) \bar{\eta}_i + \hat{g}_i(\bar{z}_i, \xi_i, \zeta_i, r_i, w) \\ \hat{g}_i &= -k_{in_i-1} k_{i1} \bar{x}_{i1} + \sum_{j=2}^{n_i-1} (k_{ij-1} - k_{in_i-1} k_{ij}) \bar{x}_{ij} \\ &\quad + k_{in_i-1} \zeta_i + \bar{g}_i(\bar{z}_i, \bar{x}_i, r_i, w) \end{aligned}$$

It can be verified that $\hat{g}_i(\mathbf{0}, \mathbf{0}, \mathbf{0}, r_i, w) = 0$, $\bar{g}_i(\mathbf{0}, \mathbf{0}, \mathbf{0}, r_i, w) = 0$, and $\check{g}_i(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, r_i, w) = 0$ for all $r_i \in \mathbb{R}$ and $w \in \mathbb{R}^{n_w}$. Denote $\bar{z}_i = \text{col}(\bar{z}_i, \xi_i)$ and $\hat{z}_i = \text{col}(\bar{z}_i, \bar{\eta}_i)$. For $n_i = 1$, the ξ_i -subsystem vanishes and we let $\bar{z}_i = \bar{z}_i$, $\zeta_i = \bar{x}_{i1}$ for consistency.

According to Lemma 11.1(iv) in [30] and by completing the square, there exist some known smooth functions $\hat{\phi}_{i1}^0, \hat{\phi}_{i2}, \hat{\phi}_{i3} > 1$ such that, for all $r_i \in \mathbb{R}$ and $w \in \mathbb{W}$,

$$\|\hat{g}_i(\bar{z}_i, \zeta_i, r_i, w)\|^2 \leq \hat{\phi}_{i1}^0(r_i, w) [\hat{\phi}_{i2}(\bar{z}_i) \|\bar{z}_i\|^2 + \hat{\phi}_{i3}(\zeta_i) \zeta_i^2] \quad (13)$$

By Lemma 11.1(i) in [30], there exist known smooth functions $\hat{\phi}_{i1}, \hat{\phi}_{i4} > 1$ and unknown constants $\hat{c}_{ig}, \hat{\ell}_{i\psi} > 1$ satisfying

$$\hat{\phi}_{i1}^0(r_i, w) \leq \hat{c}_{ig} \hat{\phi}_{i1}(r_i), \quad \psi_i^2(r_i, w) \leq \hat{\ell}_{i\psi} \hat{\phi}_{i4}(r_i) \quad (14)$$

It follows that, for all $r_i \in \mathbb{R}$ and $w \in \mathbb{W}$,

$$\|\hat{g}_i(\bar{z}_i, \zeta_i, r_i, w)\|^2 \leq \hat{c}_{ig} \hat{\phi}_{i1}(r_i) [\hat{\phi}_{i2}(\bar{z}_i) \|\bar{z}_i\|^2 + \hat{\phi}_{i3}(\zeta_i) \zeta_i^2] \quad (15)$$

Similarly, one can determine some known smooth functions $\check{\phi}_{i1}, \check{\phi}_{i2}, \check{\phi}_{i3} > 1$ and unknown constant $\check{c}_{ig} > 1$ such that, for all $r_i \in \mathbb{R}$ and $w \in \mathbb{W}$,

$$\|\check{g}_i(\hat{z}_i, \zeta_i, r_i, w)\|^2 \leq \check{c}_{ig} \check{\phi}_{i1}(r_i) [\check{\phi}_{i2}(\hat{z}_i) \|\hat{z}_i\|^2 + \check{\phi}_{i3}(\zeta_i) \zeta_i^2] \quad (16)$$

We claim the \hat{z}_i -subsystem admits the following property.

Lemma 4: For each $i \in \mathcal{N}$, let $\kappa_i(r_i) \geq \hat{\phi}_{i1}(r_i) + 1$. Then, there exists a continuously differentiable function $W_i(\hat{z}_i)$ such that, for all $r_i \in \mathbb{R}$ and $w \in \mathbb{W}$, along the trajectory of (12),

$$\begin{aligned} \underline{\hat{\alpha}}_i(\|\hat{z}_i\|) &\leq W_i(\hat{z}_i) \leq \bar{\hat{\alpha}}_i(\|\hat{z}_i\|) \\ \dot{W}_i(\hat{z}_i) &\leq -\|\hat{z}_i\|^2 + \hat{\sigma}_{i\zeta} \hat{\gamma}_{i\zeta}(\zeta_i, r_i) \zeta_i^2 + \hat{\sigma}_{i\mu} \hat{\gamma}_{i\mu}(\mu_i, r_i) \mu_i^2 \end{aligned}$$

for some known smooth functions $\hat{\alpha}_i, \hat{\bar{\alpha}}_i \in \mathcal{K}_\infty$, $\hat{\gamma}_{i\zeta}, \hat{\gamma}_{i\mu} > 1$, and unknown constants $\hat{\sigma}_{i\zeta}, \hat{\sigma}_{i\mu} > 1$.

The proof of Lemma 4 is put in **Appendix**.

Motivated by [28], [29], we let $\bar{u}_i = -\theta_i \rho_i(\zeta_i, r_i) \zeta_i$ with $\theta_i = \tau_i(\zeta_i, r_i)$. Here, ρ_i and τ_i are positive smooth functions to be specified later and θ_i is a dynamic gain to handle the unknown boundaries of static uncertainties. For simplicity, we set $\theta_i(0) = 0$. The developed partial stabilizer for the augmented system (7)–(9) is consequently

$$\begin{aligned} u_i &= -\theta_i \rho_i(\zeta_i, r_i) \zeta_i + \kappa_i(r_i) \eta_i \\ \dot{\eta}_i &= -\kappa_i(r_i) \eta_i + u_i \\ \dot{\theta}_i &= \tau_i(\zeta_i, r_i) \end{aligned} \quad (17)$$

It is of the form (10) and distributed in the sense of using each agent's own and neighboring information.

We are ready to present our main theorem.

Theorem 1: Under Assumptions 1–4, there exist positive constants α, β and smooth functions $\kappa_i(r_i), \rho_i(\zeta_i, r_i), \tau_i(\zeta_i, r_i)$ such that Problem 1 for multi-agent system (2) is solved by a distributed controller of the following form

$$\begin{aligned} u_i &= -\theta_i \rho_i(\zeta_i, r_i) \zeta_i + \kappa_i(r_i) \eta_i \\ \dot{\eta}_i &= -\kappa_i(r_i) \eta_i + u_i \\ \dot{\theta}_i &= \tau_i(\zeta_i, r_i) \\ \dot{r}_i &= -\alpha \nabla f_i(r_i) - \beta \sum_{j=1}^N a_{ij}(r_i - r_j) - \sum_{j=1}^N a_{ij}(v_i - v_j) \\ \dot{v}_i &= \alpha \beta \sum_{j=1}^N a_{ij}(r_i - r_j) \end{aligned} \quad (18)$$

Proof: Set α, β and $\kappa_i(r_i)$ as in Lemmas 2 and 4. By Lemma 3, we are left to show the following closed-loop system admits well-defined bounded trajectories for $t \geq 0$ and is globally asymptotically e_i -semistable at 0.

$$\begin{aligned} \dot{\hat{z}}_i &= \hat{h}_i(\hat{z}_i, \zeta_i, r_i, w, \mu_i) \\ \dot{\zeta}_i &= \check{g}_i(\hat{z}_i, \zeta_i, r_i, w) - \theta_i b_i(w) \rho_i(\zeta_i, r_i) \zeta_i - k_{i1} \mu_i \\ \dot{\theta}_i &= \tau_i(\zeta_i, r_i) \\ \dot{r}_i &= -\alpha \nabla f_i(r_i) - \beta \sum_{j=1}^N a_{ij}(r_i - r_j) - \sum_{j=1}^N a_{ij}(v_i - v_j) \\ \dot{v}_i &= \alpha \beta \sum_{j=1}^N a_{ij}(r_i - r_j) \end{aligned} \quad (19)$$

where function \hat{h}_i is determined by (12) and $\check{g}_i(\hat{z}_i, \zeta_i, r_i, w) \triangleq \check{g}_i(\bar{z}_i, \xi_i, \bar{\eta}_i, \zeta_i, r_i, w)$ to save notations.

The proof is divided into two steps.

Step 1: we consider the first five subsystems and seek certain disturbance attenuation performance with μ_i as its disturbance by choosing ρ_i and τ_i .

First, by Lemma 4, we apply the changing supply functions technique [31] and conclude that, for any given smooth function $\hat{\Delta}_i(\hat{z}_i) > 0$, there exists a continuously differentiable function $W_i^1(\hat{z}_i)$ such that, along the trajectory of (19),

$$\begin{aligned} \hat{\alpha}_i^1(\|\hat{z}_i\|) &\leq W_i^1(\hat{z}_i) \leq \hat{\bar{\alpha}}_i^1(\|\hat{z}_i\|) \\ \dot{W}_i^1 &\leq -\hat{\Delta}_i(\hat{z}_i) \|\hat{z}_i\|^2 + \hat{\sigma}_{i\zeta}^1 \hat{\gamma}_{i\zeta}^1(\zeta_i, r_i) \zeta_i^2 + \hat{\sigma}_{i\mu}^1 \hat{\gamma}_{i\mu}^1(\mu_i, r_i) \mu_i^2 \end{aligned}$$

for some known smooth functions $\hat{\alpha}_i^1, \hat{\bar{\alpha}}_i^1 \in \mathcal{K}_\infty$, $\hat{\gamma}_{i\zeta}^1, \hat{\gamma}_{i\mu}^1 > 1$, and unknown constants $\hat{\sigma}_{i\zeta}^1, \hat{\sigma}_{i\mu}^1 > 1$.

Second, let $V_i(\hat{z}_i, \zeta_i, \bar{\theta}_i) = \hat{\ell}_i W_i^1(\hat{z}_i) + \zeta_i^2 + \bar{\theta}_i^2$, where $\bar{\theta}_i = \theta_i - \Theta_i$ with $\Theta_i, \hat{\ell}_i > 0$ to be specified later. It is positive definite and radially unbounded, and moreover satisfies

$$\begin{aligned} \dot{V}_i &\leq -\hat{\ell}_i [\hat{\Delta}_i(\hat{z}_i) \|\hat{z}_i\|^2 - \hat{\sigma}_{i\zeta}^1 \hat{\gamma}_{i\zeta}^1(\zeta_i, r_i) \zeta_i^2 - \hat{\sigma}_{i\mu}^1 \hat{\gamma}_{i\mu}^1(\mu_i, r_i) \mu_i^2] \\ &\quad + 2\zeta_i [\check{g}_i(\hat{z}_i, \zeta_i, r_i, w) - \theta_i b_i(w) \rho_i(\zeta_i, r_i) \zeta_i - k_{i1} \mu_i] \\ &\quad + 2(\theta_i - \Theta_i) \tau_i(\zeta_i, r_i) \end{aligned}$$

Recalling inequality (16), we complete the square and have

$$\begin{aligned} \dot{V}_i &\leq -[\hat{\ell}_i \hat{\Delta}_i(\hat{z}_i) - \check{c}_{ig} \check{\rho}_{i2}(\hat{z}_i)] \|\hat{z}_i\|^2 - [2\theta_i b_i(w) \rho_i(\zeta_i, r_i) \\ &\quad - \check{\rho}_{i1}(r_i) - \hat{\ell}_i \hat{\sigma}_{i\zeta}^1 \hat{\gamma}_{i\zeta}^1(\zeta_i, r_i) - \check{c}_{ig} \check{\rho}_{i3}(\zeta_i) - 1] \zeta_i^2 \\ &\quad + [\hat{\ell}_i \hat{\sigma}_{i\mu}^1 \hat{\gamma}_{i\mu}^1(\mu_i, r_i) + k_{i1}^2] \mu_i^2 + 2(\theta_i - \Theta_i) \tau_i(\zeta_i, r_i) \end{aligned}$$

Choosing

$$\begin{aligned} \hat{\ell}_i &\geq \check{c}_{ig}, \quad \hat{\Delta}_i(\hat{z}_i) \geq \check{\rho}_{i2}(\hat{z}_i) + 1 \\ \rho_i(\zeta_i, r_i) &\geq \hat{\gamma}_{i\zeta}^1(\zeta_i, r_i) + \check{\rho}_{i1}(r_i) + \check{\rho}_{i3}(\zeta_i) + 2 \\ \tau_i(\zeta_i, r_i) &= \rho_i(\zeta_i, r_i) \zeta_i^2, \quad \Theta_i \geq \frac{1}{2b_0} \max\{\hat{\ell}_i \hat{\sigma}_{i\zeta}^1, \check{c}_{ig}\} \end{aligned} \quad (20)$$

gives $\dot{V}_i \leq -\|\hat{z}_i\|^2 - \zeta_i^2 + [\hat{\ell}_i \hat{\sigma}_{i\mu}^1 \hat{\gamma}_{i\mu}^1(\mu_i, r_i) + k_{i1}^2] \mu_i^2$. By Lemma 2 and the smoothness of $\hat{\gamma}_{i\mu}^1$, there exists a constant $c_{i\mu} > 0$ satisfying $\hat{\ell}_i \hat{\sigma}_{i\mu}^1 \hat{\gamma}_{i\mu}^1(\mu_i, r_i) + k_{i1}^2 \leq c_{i\mu}$, which further implies

$$\dot{V}_i \leq -\|\hat{z}_i\|^2 - \zeta_i^2 + c_{i\mu} \mu_i^2$$

Step 2: we show that the closed-loop system (19) admits well-defined bounded trajectories for $t > 0$ and is globally asymptotically e_i -semistable at 0.

Note that the equilibria set of (19) is specified by $\mathcal{D} = \{\hat{z} = \mathbf{0}, \zeta = \mathbf{0}, r = \mathbf{1}_N y^*, v = v^* + l_N \mathbf{1}_N\}$ with an arbitrary constant l_N . For $e_i = 0$, we set $\Theta^* = \text{col}(\Theta_1, \dots, \Theta_N)$ and verify that $\text{col}(\mathbf{0}, \mathbf{0}, \Theta^*, \mathbf{1}_N y^*, v^*)$ is an equilibrium of system (19).

From the proof of Lemma 2, we know that the function $W_0(\bar{r}, \bar{v}_2)$ defined thereof satisfies $\bar{\ell}_1 \|\text{col}(\bar{r}, \bar{v}_2)\|^2 \leq W_0(\bar{r}, \bar{v}_2) \leq \bar{\ell}_2 \|\text{col}(\bar{r}, \bar{v}_2)\|^2$ and $\dot{W}_0 \leq -\bar{\ell}_3 \|\text{col}(\bar{r}, \bar{v}_2)\|^2$ for some constants $\bar{\ell}_1, \bar{\ell}_2, \bar{\ell}_3 > 0$. Due to the Lipschitzness of Π in \bar{r} , μ_i is also Lipschitz in $\text{col}(\bar{r}, \bar{v}_2)$. Thus, there exists a constant $\bar{\ell}_4 > 0$ such that $\sum_{i=1}^N c_{i\mu} \mu_i^2 \leq \bar{\ell}_4 \bar{\ell}_3 \|\text{col}(\bar{r}, \bar{v}_2)\|^2$.

Let $V = \sum_{i=1}^N V_i + \bar{\ell}_4 V_0$ with V_0 defined in the proof of Lemma 2. The first condition in Lemma 1 is verified. Taking the time derivative of V along the trajectory of (19) gives

$$\begin{aligned} \dot{V} &\leq -\|\hat{z}\|^2 - \zeta^2 + \sum_{i=1}^N c_{i\mu} \mu_i^2 - \bar{\ell}_4 \bar{\ell}_3 \|\text{col}(\bar{r}, \bar{v}_2)\|^2 \\ &\leq -\|\hat{z}\|^2 - \zeta^2 \end{aligned}$$

This implies the second inequality in Lemma 1. Overall, the function V indeed satisfies the conditions in Lemma 1. This guarantees the trajectory's boundedness over $[0, +\infty)$ and the global asymptotic e_i -semistability of system (19) at 0. By Lemma 3, we complete the proof. ■

Remark 5: The developed optimal consensus control (18) is of a high-gain type to handle the uncertainties. The parameters and functions can be sequentially constructed. First, we choose α, β according to Lemma 2. Then, we choose κ_i according to Lemma 4. Finally, we choose ρ_i, τ_i to satisfy (20).

In some case, set \mathbb{W} or at least its boundary might be known to us. Of course, we can still use the controller (18) to solve this problem. But we can further construct a simpler controller based on the information of \mathbb{W} . To this end, it is reasonable to introduce a new assumption to replace Assumption 4.

Assumption 5: For each $i \in \mathcal{N}$, there exists a continuously differentiable function $W_{\bar{z}}(\bar{z}_i)$ such that, for all $r_i \in \mathbb{R}$ and $w \in \mathbb{W}$, along the trajectory of system (9),

$$\begin{aligned} \underline{\alpha}_i(\|\bar{z}_i\|) &\leq W_{\bar{z}}(\bar{z}_i) \leq \bar{\alpha}_i(\|\bar{z}_i\|) \\ \dot{W}_{\bar{z}} &\leq -\alpha_i(\|\bar{z}_i\|) + \gamma_{ie}(e_i)e_i^2 + \gamma_{i\mu}(r_i)\mu_i^2 \end{aligned} \quad (21)$$

for some known smooth functions $\underline{\alpha}_i, \bar{\alpha}_i, \alpha_i \in \mathcal{K}_\infty$, $\gamma_{ie}, \gamma_{i\mu} > 1$ with α_i satisfying $\limsup_{s \rightarrow 0^+} \frac{\alpha_i^{-1}(s^2)}{s} < +\infty$.

In this case, we propose a reduced-order controller:

$$\begin{aligned} u_i &= -\rho_i(\zeta_i, r_i)\zeta_i + \kappa_i(r_i)\eta_i \\ \dot{\eta}_i &= -\kappa_i(r_i)\eta_i + u_i \\ \dot{r}_i &= -\alpha \nabla f_i(r_i) - \beta \sum_{j=1}^N a_{ij}(r_i - r_j) - \sum_{j=1}^N a_{ij}(v_i - v_j) \\ \dot{v}_i &= \alpha \beta \sum_{j=1}^N a_{ij}(r_i - r_j) \end{aligned} \quad (22)$$

The optimal output consensus problem can be solved by this new controller as shown in the following theorem.

Theorem 2: Under Assumptions 1–3 and 5, there exist positive constants α, β and positive smooth functions $\kappa_i(r_i), \rho_i(\zeta_i, r_i)$ such that Problem 1 for multi-agent system (2) is solved by a distributed controller of the form (22).

Proof: The proof is similar as that of Theorem 1, and we only provide some brief arguments.

First, by similar arguments as that in the proof of Lemma 4, we can show that, for each $i \in \mathcal{N}$, there exist a smooth function $\kappa_i(r_i) > 0$ and a continuously differentiable function $W_i(\hat{z}_i)$ such that, along the trajectory of system (12),

$$\begin{aligned} \underline{\hat{\alpha}}_i(\|\hat{z}_i\|) &\leq W_i(\hat{z}_i) \leq \bar{\hat{\alpha}}_i(\|\hat{z}_i\|) \\ \dot{W}_i &\leq -\|\hat{z}_i\|^2 + \hat{\gamma}_{i\zeta}(\zeta_i, r_i)\zeta_i^2 + \hat{\gamma}_{i\mu}(\mu_i, r_i)\mu_i^2 \end{aligned} \quad (23)$$

for some known smooth functions $\underline{\hat{\alpha}}_i, \bar{\hat{\alpha}}_i \in \mathcal{K}_\infty$, $\hat{\gamma}_{i\zeta}, \hat{\gamma}_{i\mu} > 1$.

Next, we apply the changing supply functions technique to the \hat{z}_i -subsystem and conclude that, for any given smooth function $\hat{\Delta}_i(\hat{z}_i) > 0$, there exists a continuously differentiable function $W_i^1(\hat{z}_i)$ such that, along the trajectory of (12),

$$\begin{aligned} \underline{\hat{\alpha}}_i^1(\|\hat{z}_i\|) &\leq W_i^1(\hat{z}_i) \leq \bar{\hat{\alpha}}_i^1(\|\hat{z}_i\|) \\ \dot{W}_i^1 &\leq -\hat{\Delta}_i(\hat{z}_i)\|\hat{z}_i\|^2 + \hat{\gamma}_{i\zeta}^1(\zeta_i, r_i)\zeta_i^2 + \hat{\gamma}_{i\mu}^1(\mu_i, r_i)\mu_i^2 \end{aligned}$$

for some known smooth functions $\underline{\hat{\alpha}}_i^1, \bar{\hat{\alpha}}_i^1 \in \mathcal{K}_\infty$, $\hat{\gamma}_{i\zeta}^1, \hat{\gamma}_{i\mu}^1 > 1$.

Let $\hat{V}_i(\hat{z}_i, \zeta_i) = W_i^1(\hat{z}_i) + \zeta_i^2$. By Lemma 11.1 in [30] and completing the square, one can obtain that

$$\begin{aligned} \dot{\hat{V}}_i &\leq -[\hat{\Delta}_i(\hat{z}_i) - \check{\phi}_{i2}(\hat{z}_i)]\|\hat{z}_i\|^2 + [\hat{\gamma}_{i\mu}^1(\mu_i, r_i) + k_{i1}^2]\mu_i^2 \\ &\quad - [2b_i(w)\rho_i(\zeta_i, r_i) - \hat{\gamma}_{i\zeta}^1(\zeta_i, r_i) - \check{\phi}_{i1}(r_i) - \check{\phi}_{i3}(\zeta_i) - 1]\zeta_i^2 \end{aligned}$$

for some known smooth functions $\check{\phi}_{i1}, \check{\phi}_{i2}, \check{\phi}_{i3} > 1$. Letting $\hat{\Delta}_i(\hat{z}_i) \geq \check{\phi}_{i2}(\hat{z}_i) + 1$, $\rho_i(\zeta_i, r_i) \geq \frac{1}{2b_0}[\hat{\gamma}_{i\zeta}^1(\zeta_i, r_i) + \check{\phi}_{i1}(r_i) + \check{\phi}_{i3}(\zeta_i) + 2]$ implies $\dot{\hat{V}}_i \leq -\|\hat{z}_i\| - \hat{c}_{i\mu}\mu_i^2$ for some constant $\hat{c}_{i\mu} > 0$. Then, the arguments of *Step 2* in the proof of Theorem 1 proceed as well and thus complete the proof. ■

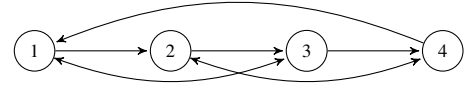


Fig. 1. Communication digraph \mathcal{G} in our examples.

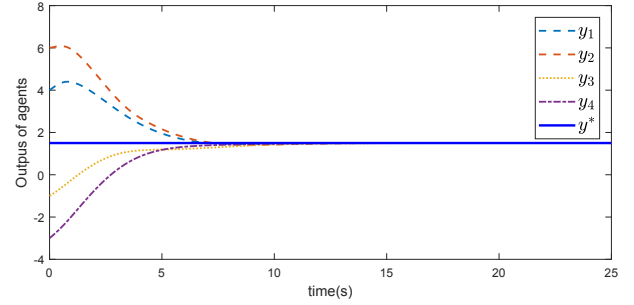


Fig. 2. Profiles of agent outputs in Example 1.

Since we are supposed to know the boundary of set \mathbb{W} , no adaptive component is needed in controller (22). In this case, the rest parameters and functions can be derived in a similar way as mentioned in Remark 5.

Remark 6: The controllers (18) and (22) are both composed of two parts constructed in two steps: optimal signal generator for problem (4) and distributed partial stabilizer for the augmented system composed of (7) and (9). By this two-step procedure and dynamic compensator based feedback designs, the technical difficulties brought by nonlinearities, uncertainties and optimal requirements are successfully overcome.

Remark 7: Compared with relevant reference [12], the multi-agent system (2) is further subject to dynamic uncertainties. Moreover, the considered agents are nonlinearly parameterized with respect to uncertainties in contrast to the linear parameterized fashion in [12]. As the pure adaptive rules fail to solve this problem, a novel robust distributed controller has been developed to deal with the complicated uncertainties.

V. SIMULATION

In this section, we present two examples to illustrate the effectiveness of our designs.

Example 1 Consider a rendezvous problem [32] for four single-link manipulators with flexible joints as follows:

$$\begin{aligned} J_{i1}\ddot{q}_{i1} + M_i g L_i \sin q_{i1} + k_i(q_{i1} - q_{i2}) &= 0 \\ J_{i2}\ddot{q}_{i2} - k_i(q_{i1} - q_{i2}) &= u_i \end{aligned} \quad (24)$$

where q_{i1}, q_{i2} are the angular positions, J_{i1}, J_{i2} are the moments of inertia, M_i is the total mass, L_i is a distance, k_i is a spring constant, and u_i is the torque input. The communication digraph among these agents is depicted as Fig. 1 with unity edge weights with $\lambda_2 = 2$ and $\lambda_4 = 3$.

To steer these manipulators to rendezvous at a common position that minimizes the aggregate distance from their starting position to this final position, we let $y_i = q_{i1}$ and take the cost functions as $f_i(y_i) = \frac{1}{2}\|y_i - q_{i1}(0)\|^2$ and $f(y) = \frac{1}{2}\sum_{i=1}^4\|y - q_{i1}(0)\|^2$ ($i = 1, \dots, 4$). One can check that the optimal solution of the global cost function is $y^* = \frac{1}{4}\sum_{i=1}^4 q_{i1}(0)$. To make this problem more interesting, we assume that $M_i = (1 + w_{i1})M_{i0}$

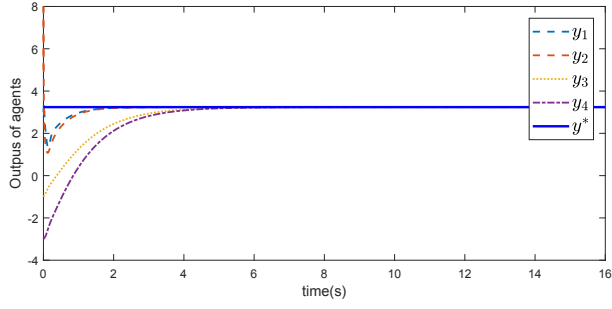


Fig. 3. Profiles of agent outputs in Example 2.

and $L_i = (1 + w_{i2})L_{i0}$ with nominal mass M_{i0} , nominal length L_{i0} , and uncertain parameters w_{i1}, w_{i2} .

Letting $x_i = \text{col}(q_{i1}, \dot{q}_{i1}, q_{i1}^{(2)}, q_{i1}^{(3)})$, we rewrite system (24) into the form (2) with $w = \text{col}(w_{11}, w_{12}, \dots, w_{41}, w_{42}), n_i = 4, b_i(w) = \frac{k_i}{J_{i1}J_{i2}}$ and $g_i(x_i, w) = -x_{i3}[\frac{M_i g_i L_i}{J_{i1}} \cos(x_{i1}) + \frac{k_i}{J_{i1}} + \frac{k_i}{J_{i2}}] + \frac{M_i g_i L_i}{J_{i1}}(x_{i2}^2 - \frac{k_i}{J_{i2}}) \sin(x_{i1})$. We can verify all assumptions in this paper and solve this problem according to Theorem 1.

For simulations, we set $J_{i1} = 1, J_{i2} = 1, L_{i0} = 1, M_i = 1, k_i = 1$ for simplicity and the uncertain parameters are randomly chosen such that $w_{i1}, w_{i2} \geq 0$. Following the procedures in Lemma 2 and Theorem 1, we select $\alpha = 1, \beta = 15$ for the generator (7) and $k_{i1} = 1, k_{i2} = 3, k_{i3} = 3, \kappa_i(r_i) = 1, \rho_i(\zeta_i, r_i) = \zeta_i^4 + 1, \tau_i(\zeta_i, r_i) = \rho_i(\zeta_i, r_i)\zeta_i^2$ for the controller (22) with $1 \leq i \leq 4$. All initial conditions are randomly chosen and the simulation result is shown in Fig. 2, where the optimal rendezvous can be observed on y^* .

Example 2 Consider another multi-agent system including two controlled FitzHugh-Nagumo dynamics [33]

$$\begin{aligned} \dot{z}_i &= -(1 + w_{i3})cz_i + (1 - w_{i4})bx_i \\ \dot{x}_i &= (1 + w_{i6})x_i(a - x_i)(x_i - 1) - z_i + (1 + w_{i5})u_i \\ y_i &= x_i, \quad i = 1, 2 \end{aligned}$$

and two controlled Van der Pol oscillators [13]

$$\begin{aligned} \dot{x}_{i1} &= x_{i2} \\ \dot{x}_{i2} &= -(1 + w_{i3})x_{i1} + (1 + w_{i4})(1 - x_{i1}^2)x_{i2} + (1 + w_{i5})u_i \\ y_i &= x_{i1}, \quad i = 3, 4 \end{aligned}$$

with input u_i , output y_i , constants $a, b, c > 0$, and unknown parameter w_{ij} . Let $w = \text{col}(w_{13}, w_{14}, \dots, w_{44}, w_{45})$. Clearly, all these agents are of the form (2).

We consider the optimal output consensus problem for this heterogeneous multi-agent system with more complicated cost functions as $f_1(y) = (y - 8)^2, f_2(y) = \frac{y^2}{80 \ln(y^2 + 2)} + (y - 5)^2, f_3(y) = \frac{y^2}{20\sqrt{y^2 + 1}} + y^2, f_4(y) = \ln(e^{-0.05y} + e^{0.05y}) + y^2$. Using the inequalities $0 \leq \frac{1}{\ln(y^2 + 2)} \leq 1.5, 0 \leq \frac{1}{\sqrt{y^2 + 1}} \leq 1, -1 \leq \frac{e^{-0.05y} - e^{0.05y}}{e^{0.05y} + e^{-0.05y}} \leq 1$, we can verify Assumption 2 with $L_i = 1$ and $\bar{l}_i = 3$ for $i = 1, \dots, 4$. Furthermore, the global optimal point is $y^* = 3.24$ by numerically minimizing $\sum_{i=1}^4 f_i(y)$.

Let $a = 0.2, b = 0.8, c = 0.8$. The uncertain parameters are randomly chosen such that $w_{i3}, w_{i5} \geq 0$ for $i = 1, \dots, 4$. Without knowing the boundary of the compact set \mathbb{W} containing these uncertainties, the controllers in [19] fail to solve the

associated optimal output consensus problem. However, we can verify Assumptions 3 and 4 for $i = 1, 2$ with $z_i^*(s, w) = \frac{(1 - w_{i2})b}{(1 + w_{i1})c}s, W_{\bar{z}}(s) = \alpha_i(s) = s^2, \gamma_{ie}(s) = \gamma_{i\mu}(s) = 1$. Note that these two assumptions trivially hold for $i = 3, 4$. According to Theorem 1, the associated optimal output consensus problem can be solved by a distributed controller of the form (18). For simulations, we still use $\alpha = 1, \beta = 15$, and then choose $\rho_i(\zeta_i, r_i) = \zeta_i^4 + r_i^4 + 1, \kappa_i(r_i) = r_i^4 + 1, \tau_i(\zeta_i, r_i) = \rho_i(\zeta_i, r_i)\zeta_i^2$ with $\zeta_i = x_i - r_i$ for $i = 1, 2$ and $\rho_i(\zeta_i, r_i) = \zeta_i^4 + r_i^4 + 1, \kappa_i(r_i) = r_i^4 + 1, \tau_i(\zeta_i, r_i) = \rho_i(\zeta_i, r_i)\zeta_i^2$ with $\zeta_i = x_{i1} - r_i + x_{i2}$ for $i = 3, 4$. All initial conditions are randomly chosen and the simulation result is shown in Fig. 2, where a satisfactory performance can be observed and the optimal output consensus is achieved on the optimal point $y^* = 3.24$.

VI. CONCLUSION

We have studied an optimal output consensus problem for a class of heterogeneous high-order nonlinear systems with both static and dynamic uncertainties. We proposed a two-step design scheme to convert it into two subproblems: optimal consensus for single-integrator multi-agent system and distributed partial stabilization of some augmented nonlinear systems. By adding a dynamic compensator to deal with the uncertainties, we constructed two distributed controls for this problem under standing conditions. Our future works include the MIMO extension with time-varying digraphs.

APPENDIX. PROOF OF LEMMA 4

The proof is completed by successively using the changing supply functions technique [31].

We first consider the case when $n_i \geq 2$. Under Assumption 4, we apply the changing supply functions technique to the \bar{z}_i -subsystem and conclude that, for any given $\bar{\Delta}_{\bar{z}}(\bar{z}_i) > 0$, there exists a continuously differentiable function $W_{\bar{z}}^1(\bar{z}_i)$ satisfying $\underline{\alpha}_{\bar{z}}(\|\bar{z}_i\|) \leq W_{\bar{z}}^1(\bar{z}_i) \leq \bar{\alpha}_{\bar{z}}(\|\bar{z}_i\|)$ and

$$\dot{W}_{\bar{z}}^1 \leq -\bar{\Delta}_{\bar{z}}(\bar{z}_i)\|\bar{z}_i\|^2 + \bar{\sigma}_{i\xi}\bar{\gamma}_{i\xi}^1(\xi_i)\|\xi_i\|^2 + \bar{\sigma}_{i\mu}\bar{\gamma}_{i\mu}^1(\mu_i, r_i)\mu_i^2$$

for some known smooth functions $\underline{\alpha}_{\bar{z}}, \bar{\alpha}_{\bar{z}} \in \mathcal{K}_\infty, \bar{\gamma}_{i\xi}^1, \bar{\gamma}_{i\mu}^1 > 1$ and unknown constants $\bar{\sigma}_{i\xi}, \bar{\sigma}_{i\mu} > 1$.

From the choice of k_{ij} , matrix A_i^0 is Hurwitz. Then, there exists a unique positive definite matrix \bar{P}_i satisfying $A_i^{0T}\bar{P}_i + \bar{P}_i A_i^0 = -3I_{m-1}$. Let $W_{i\xi}^0(\xi_i) = \xi_i^T \bar{P}_i \xi_i$. Its time derivative along the trajectory of (9) satisfies

$$\begin{aligned} \dot{W}_{i\xi}^0 &= 2\xi_i^T \bar{P}_i [A_i^0 \xi_i + B_i^0 \zeta_i - E_i^0 \mu_i] \\ &\leq -\|\xi_i\|^2 + \|\bar{P}_i B_i^0\|^2 \|\zeta_i\|^2 + \|\bar{P}_i E_i^0\|^2 \mu_i^2 \end{aligned}$$

By changing supply functions of ξ_i -subsystem, for any given $\bar{\Delta}_{i\xi}(\xi_i) > 0$, there exists a continuously differentiable function $W_{i\xi}^1(\xi_i)$ satisfying $\underline{\alpha}_{i\xi}(\|\xi_i\|) \leq W_{i\xi}^1(\xi_i) \leq \bar{\alpha}_{i\xi}(\|\xi_i\|)$ and

$$\dot{W}_{i\xi}^1 \leq -\bar{\Delta}_{i\xi}(\xi_i)\|\xi_i\|^2 + \bar{\gamma}_{i\xi}(\zeta_i)\|\zeta_i\|^2 + \bar{\gamma}_{i\mu}(\mu_i)\mu_i^2$$

for some known smooth functions $\underline{\alpha}_{i\xi}, \bar{\alpha}_{i\xi} \in \mathcal{K}_\infty, \bar{\gamma}_{i\xi}, \bar{\gamma}_{i\mu} > 1$.

Let $W_{\bar{z}}^1(\bar{z}_i) = W_{\bar{z}}^1(\bar{z}_i) + \bar{\sigma}_{i\xi} W_{i\xi}^1(\xi_i)$. Clearly, there exist functions $\underline{\alpha}_{\bar{z}}, \bar{\alpha}_{\bar{z}} \in \mathcal{K}_\infty$ satisfying $\underline{\alpha}_{\bar{z}}(\|\bar{z}_i\|) \leq W_{\bar{z}}^1(\bar{z}_i) \leq \bar{\alpha}_{\bar{z}}(\|\bar{z}_i\|)$. Its time derivative along the trajectory of (12) satisfies

$$\dot{W}_{\bar{z}}^1 \leq -\bar{\Delta}_{\bar{z}}(\bar{z}_i)\|\bar{z}_i\|^2 - \bar{\sigma}_{i\xi}(\bar{\Delta}_{i\xi}(\xi_i) - \bar{\gamma}_{i\xi}^1(\xi_i))\|\xi_i\|^2$$

$$+ \bar{\sigma}_{i\xi} \bar{\gamma}_{i\xi}(\zeta_i) \|\zeta_i\|^2 + \bar{\sigma}_{i\xi} \bar{\gamma}_{i\mu}(\mu_i) \mu_i^2 + \bar{\sigma}_{i\mu} \bar{\gamma}_{i\mu}^1(\mu_i, r_i) \mu_i^2$$

Letting $\bar{\Delta}_{i\xi}(\bar{z}_i) > 1$, $\bar{\Delta}_{i\xi}(\xi_i) > \bar{\gamma}_{i\xi}^1(\xi_i) + 1$, $\bar{\gamma}_{i\mu}(\mu_i, r_i) > \bar{\gamma}_{i\mu}^1(\mu_i) + \bar{\gamma}_{i\mu}^1(\mu_i, r_i)$, and $\bar{\sigma}_{i\mu} > \max\{\bar{\sigma}_{i\xi}, \bar{\sigma}_{i\mu}\}$ gives

$$\dot{W}_{i\bar{z}} \leq -\|\bar{z}_i\|^2 + \bar{\sigma}_{i\xi} \bar{\gamma}_{i\xi}(\zeta_i) \|\zeta_i\|^2 + \bar{\sigma}_{i\mu} \bar{\gamma}_{i\mu}(\mu_i, r_i) \mu_i^2$$

When $n_i = 1$, the above property trivially holds for $\bar{z}_i = \bar{z}_i$.

Next, we apply the changing supply functions technique to \bar{z}_i -subsystem and conclude that, for any given smooth function $\bar{\Delta}_i(\bar{z}_i) > 0$, there exists a continuously differentiable function $W_{i\bar{z}}^1(\bar{z}_i)$ satisfying $\bar{\alpha}_i^1(\|\bar{z}_i\|) \leq W_{i\bar{z}}^1(\bar{z}_i) \leq \bar{\alpha}_i^1(\|\bar{z}_i\|)$ and

$$\dot{W}_{i\bar{z}}^1 \leq -\bar{\Delta}_i(\bar{z}_i) \|\bar{z}_i\|^2 + \bar{\sigma}_{i\xi} \bar{\gamma}_{i\xi}^1(\zeta_i) \|\zeta_i\|^2 + \bar{\sigma}_{i\mu} \bar{\gamma}_{i\mu}^1(\mu_i, r_i) \mu_i^2$$

for some known smooth functions $\bar{\alpha}_i^1, \bar{\alpha}_i^1 \in \mathcal{K}_\infty$, $\bar{\gamma}_{i\xi}^1, \bar{\gamma}_{i\mu}^1 > 1$, and unknown constants $\bar{\sigma}_{i\xi}, \bar{\sigma}_{i\mu} > 1$.

Let $W_i(\hat{z}_i) = \bar{\ell}_i W_{i\bar{z}}^1(\bar{z}_i) + \bar{\eta}_i^1 \bar{\eta}_i$ with $\bar{\ell}_i > 0$ to be specified later. Clearly, the first inequality in Lemma 4 holds. We take time derivative of W_i along the trajectory of (19) and have

$$\begin{aligned} \dot{W}_i \leq & -\bar{\ell}_i [\bar{\Delta}_i(\bar{z}_i) \|\bar{z}_i\|^2 - \bar{\sigma}_{i\xi} \bar{\gamma}_{i\xi}^1(\zeta_i) \|\zeta_i\|^2 - \bar{\sigma}_{i\mu} \bar{\gamma}_{i\mu}^1(\mu_i, r_i) \mu_i^2] \\ & + 2\bar{\eta}_i^1 [-\kappa_i(r_i) \bar{\eta}_i + \bar{g}_i(\bar{z}_i, \xi_i, \zeta_i, r_i, w) + \psi_i(r_i, w) \mu_i] \end{aligned}$$

Jointly with the inequalities (13) and (14), we can bound the cross terms by completing the square and have

$$\begin{aligned} \dot{W}_i \leq & -[\bar{\ell}_i \bar{\Delta}_i(\bar{z}_i) - \frac{2\hat{c}_{i\xi} \hat{\phi}_{i2}(\bar{z}_i)}{b_0^2}] \|\bar{z}_i\|^2 \\ & - [\kappa_i(r_i) - \frac{\hat{\phi}_{i1}(r_i)}{2} - \frac{1}{2}] \|\bar{\eta}_i\|^2 \\ & + [\bar{\ell}_i \bar{\sigma}_{i\xi} \bar{\gamma}_{i\xi}^1(\zeta_i) + \frac{\kappa_i(r_i)}{b_0^2} + \frac{2\hat{c}_{i\xi} \hat{\phi}_{i3}(\zeta_i)}{b_0^2}] \|\zeta_i\|^2 \\ & + [2\bar{\ell}_i \bar{\psi} \hat{\phi}_{i4}(r_i) + \bar{\ell}_i \bar{\sigma}_{i\mu} \bar{\gamma}_{i\mu}^1(\mu_i, r_i)] \mu_i^2 \end{aligned}$$

Note that $\kappa_i(r_i) \geq \hat{\phi}_{i1}(r_i) + 1$. Letting $\bar{\ell}_i > \frac{2\hat{c}_{i\xi}}{b_0^2} + 1$, $\bar{\Delta}_i(\bar{z}_i) > \hat{\phi}_{i2}(\bar{z}_i) + 1$, $\bar{\sigma}_{i\xi} > \bar{\ell}_i \bar{\sigma}_{i\xi} + \frac{2\hat{c}_{i\xi}}{b_0^2}$, $\bar{\sigma}_{i\mu} > \bar{\ell}_i \bar{\sigma}_{i\mu} + 2\bar{\ell}_i \bar{\psi}$, $\bar{\gamma}_{i\xi}^1(\zeta_i, r_i) > \bar{\gamma}_{i\xi}^1(\zeta_i) + \kappa_i(r_i) + \hat{\phi}_{i3}(\zeta_i)$, and $\bar{\gamma}_{i\mu}^1(\mu_i, r_i) > \hat{\phi}_{i4}(r_i) + \bar{\gamma}_{i\mu}^1(\mu_i, r_i)$ implies the second inequality and thus completes the proof.

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