Abstract—In this article, we investigate an optimal output consensus problem for heterogeneous uncertain nonlinear multiagent systems. The considered agents are described by high-order nonlinear dynamics subject to both static and dynamic uncertainties. A two-step design, comprising sequential constructions of optimal signal generator and distributed partial stabilization feedback controller, is developed to overcome the difficulties brought by nonlinearities, uncertainties, and optimal requirements. This article can not only assure an output consensus, but also achieve an optimal agreement characterized by a distributed optimization problem.

Index Terms—Adaptive control, distributed optimization, multiagent system, optimal output consensus, uncertainties.

I. INTRODUCTION

In the past few years, distributed optimization has attracted much attention due to its broad potential applications in multirobot systems, smart grid, and sensor networks. In a typical setting, each agent has access to a private objective function and all agents are regulated to achieve a consensus on the optimal solution of the sum of all local functions. Many important results were obtained based on gradients or subgradients of the local objective functions combined with consensus rules, including both discrete-time and continuous-time algorithms [1]–[6].

Since distributed optimization tasks may be implemented or depend on physical dynamics in practice, optimal consensus involving high-order agent dynamics deserves further investigation. Compared with the pure (output) consensus problem, the consensus point for all outputs of agents is additionally required to be an optimal solution of the global cost function. Note that this optimal solution can only be determined and reached in a distributed way. Some interesting attempts have been made in [7]–[9] for integrator agents, [10] for linear agents, and [11], [12] for special classes of nonlinear agents. However, optimal output consensus for more general nonlinear multiagent systems is still far from being solved, especially for agents being heterogeneous and subject to uncertainties.

In this article, we consider nonlinear multiagent systems in the Byrnes–Isidori normal form, which can model many typical mechanical and electromechanical systems [13]. In the literature, there have been many consensus results for agents of this type (e.g., [14]–[16]). This normal form is general enough to cover the dynamics reported in existing optimal consensus results [7]–[12], [17], [18]. Here, we further take into account heterogeneous nonlinear dynamics having both static and dynamic uncertainties, which inevitably bring technical difficulties in resolving the optimal output consensus problem. In a preliminary work [19], this problem was studied for such class of agents assuming that the compact set containing static uncertainties is prior known. In this article, we remove such restrictive condition and allow the boundary of this compact set to be unknown.

The contribution of this article is at least twofold. First, we solve the optimal output consensus problem for a larger class of uncertain nonlinear multiagent systems, significantly improving the existing results reported in [9]–[12]. Second, a novel dynamic compensator based distributed controller is developed for effectively addressing complicated uncertainties, while the precise information of system dynamics is required in [7]–[9]. Moreover, in contrast with relevant results in [11] and [19], the boundary of the compact set containing uncertain parameters is allowed to be unknown.

The rest of this article is organized as follows. Preliminaries and problem formulation are presented in Section II. Then, the design scheme and main results are provided in Sections III and IV with an illustrative example in Section V. Finally, conclusions are given in Section VI.

Notation: Let $\mathbb{R}^N$ be the $N$-dimensional Euclidean space. Denote col($a_1$, $\ldots$, $a_N$) = $[a_1^\top, \ldots, a_N^\top]^\top$ for vectors $a_1$, $\ldots$, $a_N$. $I_N$ (or $0_N$) denotes an all-one (or all-zero) vector in $\mathbb{R}^N$ and $I_N$ denotes the $N \times N$ identity matrix. Let $M_1 = \frac{1}{N} 1_N$, and $M_2$ be the matrices satisfying $M_2^\top M_1 = 0_{N,1}$, and $M_2^\top M_2 = I_{N,1}$, and $M_2^\top M_1 = I_N - M_1 M_1^\top$. Denote the Euclidean norm of vector $a$ by $\|a\|$ and the spectral norm of matrix $A$ by $\|A\|$. A continuous function $\alpha : [0, +\infty) \to [0, +\infty)$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It further belongs to class $\mathcal{K}_\infty$ if it belongs to class $\mathcal{K}$ and $\lim_{s\to+\infty} \alpha(s) = \infty$.

II. PRELIMINARIES AND PROBLEM FORMULATION

In this section, we present preliminaries of partial stability and graph theory, and then the formulation of our problem.

A. Partial Stability

To achieve optimal output consensus, we need to ensure the convergence of particular partial state of the closed-loop system rather than the full state. Such an issue is often referred to as partial stability (stabilization) [20]. Since the closed-loop system may have a continuum of equilibria, we introduce a modified version of partial stability as follows.
Consider the nonlinear autonomous system

\[ \dot{x}_1 = f_1(x_1, x_2), \quad \dot{x}_2 = f_2(x_1, x_2) \]

(1)

where \( x = (x_1, x_2) \) with \( x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2} \) and the functions \( f_1, f_2 \) are sufficiently smooth. Denote the equilibria set as \( \mathcal{D} \triangleq \{ x \mid f_1(x_1, x_2) = 0, f_2(x_1, x_2) = 0 \} \).

**Definition 1:** System (1) is Lyapunov semistable with respect to \( x_1 \) (or briefly, \( x_1 \)-semistable) at \( x_1^* \) if, for every \( \epsilon > 0 \), there exist \( x_1^* \) and \( \delta > 0 \) such that \( x_1^* = \text{col}(x_1^*, x_2^*) \in \mathcal{D} \) and \( ||x(0) - x_1^*|| < \delta \) implies \( ||x(t) - x_1^*|| \leq \epsilon \) for all \( t \geq 0 \). If for any \( x(0) \), it further holds that \( \lim_{t \to +\infty} ||x(t) - x_1^*|| = 0 \), this system is globally asymptotically \( x_1 \)-semistable at \( x_1^* \).

When \( \mathcal{D} = \{ 0 \} \), this definition is exactly the partial stability concept with respect to \( x_1 \) specified in [20, p. 17]. The following lemma is slightly modified from [21, Ths. 4.5 and 4.7] and its proof is omitted.

**Lemma 1:** Suppose that there exist a continuously differentiable function \( V(x) \) and a constant vector \( \chi^* \in \mathbb{R}^{n_2} \) such that \( x^* = \text{col}(x_1^*, x_2^*) \in \mathcal{D} \), and along the trajectory of (1),

\[ \alpha(||x - x^*||) \leq V(x) \leq \beta(||x - x^*||) \]

\[ \dot{V}(x) \leq -\gamma(||x_1 - x_1^*||) \]

for some functions \( \alpha, \beta \in K_{\infty} \) and \( \gamma \in \mathcal{K} \). Then, system (1) admits well-defined bounded trajectories over \([0, +\infty)\) and is globally asymptotically \( x_1 \)-semistable at \( x_1^* \).

**B. Graph Notion**

A weighted directed graph (digraph) is described by \( G = (\mathcal{N}, \mathcal{E}, A) \) with node set \( \mathcal{N} = \{ 1, \ldots, N \} \) and edge set \( \mathcal{E} \). Denote the static uncertainties is not necessarily known here. It is assumed that all functions are sufficiently smooth and satisfy \( b_i(0, 0, w) = 0, g_i(0, 0, w) = 0 \) for all \( w \in \mathbb{W} \) with some constant \( b_0 \).

We endow each agent output with a local cost function \( f_i : \mathbb{R} \to \mathbb{R} \), and define the global cost function as the sum of all local costs, i.e.,

\[ f(y) = \sum_{i=1}^N f_i(y_i) \]

for multigraph system (2), we aim to develop an algorithm such that all agent outputs achieve a consensus on the minimizer to this global cost function in a distributed fashion. For this purpose, a digraph \( G = (\mathcal{N}, \mathcal{E}, A) \) is used to describe the information communication relationships among agents with node set \( \mathcal{N} = \{ 1, \ldots, N \} \), edge set \( \mathcal{E} \subset \mathcal{N} \times \mathcal{N} \), and weighted matrix \( A \in \mathbb{R}^{n \times n} \). An edge \((j, i) \in \mathcal{E} \) with weight \( a_{ij} > 0 \) means that agent \( i \) can get the information of agent \( j \).

The considered distributed controller is described by

\[ u_i = \Xi_{i1}(\nabla f_i, x_j, y_j, j \in \mathcal{N}_{0i}) \]

\[ \chi_i = \Xi_{i2}(\nabla f_i, x_j, y_j, j \in \mathcal{N}_{0i}) \]

where \( \chi_i \in \mathbb{R}^{n_1} \) is the compensator state and \( \Xi_{i1}, \Xi_{i2} \) are smooth functions to be specified later. With these preparations, we formulate our problem explicitly as follows.

**Problem 1:** For multiagent system (2), function \( f_i, \) digraph \( G, \) and compact set \( \mathbb{W} \), find a controller of the form (3) such that, for each \( w \in \mathbb{W} \), the time derivative of the cost function defined as

\[ \frac{d}{dt} \sum_{i=1}^N \left( \frac{1}{2} \left| |y_i(t)| - y^*_i \right|^2 \right) \]

is bounded over \([0, +\infty)\).

For this agreement on the optimal point \( y^*_i \) specified by minimizing a cost function, in this sense, we say these agents achieving an optimal output consensus as in [2], [8], and [9].

This problem for single integrators has been coined as distributed optimization and investigated for many years. For high-order nonlinear agents, it is certainly more challenging to achieve such an optimal output consensus, while the static and dynamic uncertainties bring extra technical difficulties in resolving this problem.

**III. TWO-STEP DESIGN SCHEME**

In this section, we convert the optimal output consensus problem into a distributed partial stabilization problem by constructing optimal signal generators, giving rise to a two-step design scheme for solving Problem 1.

To begin with, several standing assumptions are listed.

**Assumption 1:** The digraph \( G \) is weight-balanced and strongly connected.

**Assumption 2:** For each \( i \in \mathcal{N}, \) the function \( f_i \) is twice continuously differentiable and satisfies that \( L_1 \leq \nabla^2 f_i(s) \leq L_2 \) for all \( s \in \mathbb{R}. \)
Assumption 3: For each \( i \in \mathcal{N} \), there exists a smooth function \( z_i^*(s, w) \) satisfying \( z_i^*(0, w) = 0 \) and \( h_i(z_i^*(s, w), w, s, u) = 0 \) for all \( s \in \mathbb{R} \) and \( w \in \mathbb{R}^n_u \).

Assumption 1 guarantees that each agent’s information can be reached by any other agent. Assumption 2 implies the existence and uniqueness of optimal solution to problem (4) [23]. Assumption 3 can be interpreted as the solvability of regulator equations in the context of output regulation [24]. These assumptions have been widely used (distributed) coordination for multiagent systems [3], [7], [11], [25], [26].

Consider an optimal consensus problem for a group of single integrators with the same optimal requirement (4)

\[
\dot{r}_i = \mu_i. \tag{5}
\]

If this auxiliary problem is solved by some chosen \( \mu_i \), we only need to drive agent \( i \) to track the generated signal \( r_i(t) \) to achieve the optimal output consensus for agent (2).

Since the Laplacian \( L \) of digraph \( G \) is asymmetric, the generator in [10] fails to reproduce \( y^* \) without the information of \( L \). Motivated by [18], we present a candidate of optimal signal generator for problem (4) as follows:

\[
\mu_i = -\alpha \nabla f_i(r_i) - \beta \sum_{j=1}^{N} a_{ij}(r_i - r_j) - \sum_{j=1}^{N} a_{ij}(v_i - v_j)
\]

\[
\dot{v}_i = \alpha \beta \sum_{j=1}^{N} a_{ij}(r_i - r_j) \tag{6}
\]

where \( \alpha, \beta \) are constants to be specified later. Putting it into a compact form gives

\[
\dot{r} = -\alpha \nabla \tilde{f}(r) - \beta L r - L v, \quad \dot{v} = \alpha \beta L r \tag{7}
\]

where \( r = \col(r_1, \ldots, r_N), v = \col(v_1, \ldots, v_N) \), and function \( \tilde{f}(r) \triangleq \sum_{i=1}^{N} f_i(r_i) \), is \( L \)-strongly convex while its gradient \( \nabla \tilde{f}(r) \) is \( L \)-Lipschitz with \( L = \max_{i} [l_i] \) and \( l_i = \min_{i} [l_i] \).

Let \( \tilde{r}^* \) and \( \tilde{v}^* \) be the equilibrium point of system (7). It is verified that \( \tilde{r}^* = 1_N y^* \) under Assumptions 1 and 2 by [23, Th. 3.27]. For (7), we have the following interesting result.

Lemma 2: Suppose Assumptions 1–2 hold and let \( \alpha \geq \max \{1, \frac{\beta}{\lambda} \} \), \( \beta \geq \max \{1, \frac{\lambda}{\alpha} \} \). Then, system (7) admits well-defined bounded trajectories over \([0, +\infty)\) and is globally asymptotically \( r \)-semistable at 0. Moreover, \( r_i(t) \) approaches \( y^* \) exponentially as \( t \to +\infty \) for \( i \in N \).

Proof: Briefly, we utilize Lemma 1 to complete the proof. Let \( M_L = M_1^2 M_2 \) and \( v^* = -\alpha M_2 M_1^1 \nabla \tilde{f}(r^*) \). It can be verified that \( \tilde{r}^* = (r^*, v^*) \) is an equilibrium of system (7):

\[
\dot{r}_i = -\alpha M_1 \frac{\partial f_i}{\partial r_i} \tag{8}
\]

\[
\dot{r}_2 = -\alpha M_2 \frac{\partial f_i}{\partial r_i} - \beta M_i \nabla \tilde{f}(r_i) + \alpha M_2 \nabla \tilde{f}(r_i) - \alpha M_2 \nabla \tilde{f}(r_i) \tag{9}
\]

where \( \frac{\partial f_i}{\partial r_i} \) and \( \frac{\partial f_i}{\partial r_i} \) are defined in this new coordinate with \( \alpha > 0 \) to be specified later. The first inequality in Lemma 1 apparently holds. On the other hand, by Young’s inequality, the time derivative of \( \tilde{V}_0 \) along the trajectory of (7) satisfies

\[
\dot{\tilde{V}}_0 = -2\alpha \langle \tilde{r} - \tilde{r}^* \rangle^2 + 2\beta \lambda \tilde{V}_0 + 2\alpha \lambda \tilde{V}_0 - 2\lambda \tilde{V}_2 \tag{10}
\]

\[
\leq -2\alpha \tilde{V}_0^{\beta} ||\tilde{r}||^2 - 2\beta \lambda \tilde{V}_2 ||\tilde{r}||^2 + 2\alpha \lambda \tilde{V}_2 + 2\lambda \tilde{V}_2 \tag{11}
\]

\[
\leq -2\alpha \tilde{V}_0^{\beta} ||\tilde{r}||^2 + \frac{\lambda}{\alpha} \tilde{V}_0 ||\tilde{r}||^2 + \frac{\beta}{\lambda} \tilde{V}_2 ||\tilde{r}||^2 \tag{12}
\]

\[
\leq -2\alpha \tilde{V}_0^{\beta} ||\tilde{r}||^2 + \frac{\lambda}{\alpha} \tilde{V}_0 ||\tilde{r}||^2 + \frac{\beta}{\lambda} \tilde{V}_2 ||\tilde{r}||^2 \tag{13}
\]

\[
\leq -2\alpha \tilde{V}_0^{\beta} ||\tilde{r}||^2 - \frac{1}{\alpha} \tilde{V}_0 ||\tilde{r}||^2 \tag{14}
\]

Thus, it follows that \( \tilde{V}_0 \) is exponentially stable. It can be verified that \( \tilde{V}_2 \) is exponentially stable. Thus, the system is exponentially stable. As a result, the system is globally asymptotically \( r \)-semistable at 0.

Remark 2: The optimal signal generator (7) is a modified version of the augmented Lagrangian method solving problem (4) in [18]. Here, we add an extra parameter \( \alpha \) to simplify both the synthesis and its analysis. Compared with the results for digraphs in [8], [9], and [17], our algorithm is initialization-free to generate the optimal point \( y^* \). This makes it possible to work in a scalable manner, which might be favorable for dynamic networks with leaving-off and plugging-in of agents.

Remark 3: In our design, we use the knowledge of \( \lambda_2 \) and \( \lambda_N \) as that in [7] and [25] to compensate the asymmetry of directed information flows. It should be mentioned that these values can be computed by existing algorithms beforehand, e.g., [27].

Under Assumption 3, we denote \( x_i^*(r_i) = \col(r_i, 0, x_i, \ldots, x_i, a_i, \ldots, a_i) \), \( u_i^*(r_i, w) = -\frac{\partial f_i(r_i, w)}{\partial r_i} \), and perform the coordinate transformation: \( \tilde{x}_i = \tilde{z}_i - x_i^*(r_i, w) \), \( \tilde{x}_i = x_i - x_i^*(r_i, w) \). This leads to an interconnected error system as follows:

\[
\tilde{x}_i = \tilde{x}_i - \tilde{e}_i = \tilde{x}_i - x_i - x_i^*(r_i, w) = -E_i \tilde{x}_i \tag{15}
\]
Proof: Under the lemma condition, we can confirm that trajectories of all agents are well-defined bounded over $[0, +\infty)$ and $\lim_{t \to +\infty} e_i(t) = 0$ for any initial condition $\text{col}(\tau_i(0), \tilde{\tau}_i(0), \chi^0_i(0), (r_i, 0), (v_i(0)))$. Note that $|y_i(t) - y_i^*| \leq |e_i(t)| + |r_i(t) - y_i^*|$ by the triangle inequality. This together with Lemma 2 ensures that $\lim_{t \to +\infty} |y_i(t) - y_i^*| = 0$.

Remark 4: Based on Lemma 3, our optimal output consensus problem for multiagent system (2) is converted into a distributed partial stabilization problem of certain interconnected augmented systems. As the considered nonlinear multiagent system (2) is further subject to static and dynamic uncertainties, the associated partial stabilization design problem is more challenging than relevant results obtained in [9]–[12]. On the other hand, existing designs presented in [20] and [21] are not applicable for such complicated uncertainties and the partial stabilization problem itself is nontrivial even for a single nonlinear system. Thus, we have to seek a robust distributed partial stabilization design method for the augmented systems.

IV. MAIN RESULT

In this section, we focus on the subsequent partial stabilization problem of the augmented system composed of (7) and (9) and eventually solve the optimal output consensus problem for multiagent system (2).

To this end, we make an extra assumption imposing a mild minimum-phase condition widely used in nonlinear stabilization problems [26], [28], [29].

Assumption 4: For each $i \in \mathcal{N}$, there exists a continuously differentiable function $W_i(\tilde{\tau}_i)$ such that, for all $r_i \in \mathbb{R}$ and $w \in \mathbb{W}$, along the trajectory of system (9)

$$
\omega_i(|\tilde{\tau}_i|) \leq W_i(\tilde{\tau}_i) \leq \omega_i(|\tilde{\tau}_i|)
$$

$$
W_i \leq -\alpha_i(|\tilde{\tau}_i|) + \sigma_{\text{inv}}(e_i \gamma_i w_i) + \sigma_{\text{inv}} w_i^2
$$

(11)

for some known smooth functions $\omega_i$, $\sigma_{\text{inv}}$, $\alpha_i \in \mathbb{K}_\infty$, $\gamma_i > 0$, and unknown constants $\sigma_{\text{inv}}$, $\sigma_{\text{inv}} w_i > 1$ with $\alpha_i$ satisfying $\lim_{\tilde{\tau}_i \to 0} \frac{\omega_i(\tilde{\tau}_i)}{|\tilde{\tau}_i|} = -\infty$.

Due to the presence of uncertain parameter $w$, the feedforward term $u_i^*(r_i, w_i)$ is unavailable for feedback. To tackle this issue, we introduce a dynamic compensator as follows:

$$
h_i = -\kappa_i(r_i) h_i + u_i
$$

where $\kappa_i(r_i) > 0$ is a smooth function to be specified later. Here, $\kappa_i(r_i)$ is a scaling factor to handle nonlinear functions of $r_i$. This compensator reduces to an internal model when $\kappa_i(r_i)$ is constant [24].

Consider the error system (9). For $n_i \geq 2$, choose constants $k_{ij}$ such that the polynomial $p_i(k) = \sum_{j=0}^{n_i-1} k^j + k^{n_i-1}$ is Hurwitz. Let $\xi_i = \text{col}(\tau_{i1}, \ldots, \tau_{in_i-1})$, $\xi_i = \sum_{j=1}^{n_i-1} k_{ij} \tau_{ij} + \tau_{in_i}$, and $\beta_i(r_i, r_i) \triangleq \kappa_i(r_i) \gamma_i w_i$. Performing coordinate and input transformations: $\tilde{\xi}_i = \tau_{i1} - \frac{u_i^*(r_i, w_i)}{\gamma_i w_i} - \beta_i(r_i, r_i)$ and $\tilde{\eta}_i = \eta_i - \beta_i(r_i, r_i)$ gives a composite system in the following form:

$$
\begin{align*}
\dot{z}_i &= \tilde{h}_i(\tau_i, \xi_i, r_i, w) - \frac{\partial u_i^*}{\partial r_i} \mu_i,
\dot{\zeta}_i &= A_i^o \tilde{\xi}_i + B_i^o \tilde{\eta}_i - E_i^o \mu_i,
\dot{\tilde{h}}_i &= -\kappa_i(r_i) \tilde{\eta}_i + \tilde{h}_i(\tau_i, \xi_i, r_i, w) + \psi_i(r_i, w) \mu_i
\end{align*}
$$

(12)

where

$$
A_i^o = \begin{bmatrix}
0_{n_i-2} & -I_{n_i-2} & \ldots & -I_{n_i-2}
\end{bmatrix},
B_i^o = \begin{bmatrix}
0_{n_i-2}
\end{bmatrix},
E_i^o = \begin{bmatrix}
1
\end{bmatrix}
$$

$$
\tilde{h}_i = \gamma_i \tau_i(\xi_i, r_i)
$$

(13)

By [30, Lemma 11.1(i)], there exist some known smooth functions $\phi_{01}, \phi_{02}, \phi_{03} > 1$ such that, for all $r_i \in \mathbb{R}$ and $w \in \mathbb{W}$,

$$
||\hat{g}_i(\tilde{z}_i, \xi_i, r_i, w)||^2 \leq \phi_{01}(r_i, w) ||\hat{g}_2(\tilde{z}_i)||^2 + \phi_{03}(\xi_i) \gamma_i^2.
$$

(14)

By [30, Lemma 11.1(i)], there exist some known smooth functions $\phi_{01}, \phi_{02}, \phi_{03} > 1$ such that, for all $r_i \in \mathbb{R}$ and $w \in \mathbb{W}$,

$$
||\hat{g}_i(\tilde{z}_i, \xi_i, r_i, w)||^2 \leq \phi_{02}(r_i) ||\phi_{02}(\tilde{z}_i)||^2 + \phi_{03}(\xi_i) \gamma_i^2.
$$

(15)

We claim the $\tilde{z}_i$-subsystem admits the following property.

Lemma 4: For each $i \in \mathcal{N}$, $\kappa_i(r_i) \geq \phi_{01}(r_i) + 1$. Then, there exists a continuously differentiable function $W_i(\tilde{z}_i)$ such that, for all $r_i \in \mathbb{R}$ and $w \in \mathbb{W}$, along the trajectory of (12)

$$
\begin{align*}
\tilde{h}_i(\tilde{z}_i) &\leq W_i(\tilde{z}_i) \\
W_i(\tilde{z}_i) &\leq -||\tilde{z}_i||^2 + \sigma_{\text{inv}}(\gamma_i \tau_i(\xi_i, r_i)) \gamma_i^2 + \sigma_{\text{inv}} \gamma_i w_i(\mu_i, r_i) \gamma_i^2
\end{align*}
$$

(16)

for some known smooth functions $\tilde{h}_i$, $\sigma_{\text{inv}} \in \mathbb{K}_\infty$, $\gamma_i \tau_i > 1$, and unknown constants $\sigma_{\text{inv}} \gamma_i w_i > 1$.

The proof of Lemma 4 is put in the Appendix.

Motivated by [28], [29], we let $\Xi_i = -\theta \rho_i(\xi_i, r_i) \gamma_i$ with $\theta_i = \tau_i(\xi_i, r_i)$. Here, $\rho_i$ and $\tau_i$ are positive smooth functions to be specified later and $\theta_i$ is a dynamic gain to handle the unknown boundaries of static uncertainties. For simplicity, we set $\theta_i(0) = 0$. The developed partial stabilizer for the augmented system (7)–(9) is consequently

$$
u_i = -\theta_i \rho_i(\xi_i, r_i) \gamma_i + \kappa_i(r_i) h_i
$$

$$
h_i = -\kappa_i(r_i) h_i + u_i
$$

(17)

It is of the form (10) and distributed in the sense of using each agent’s own and neighboring information.

We are ready to present our main theorem.
The developed optimal consensus control (18) is of a form that Problem 1 for multigain system (2) is solved by a distributed controller of the following form:

$$u_i = -\theta_i\rho_i(\zeta_i, r_i)\zeta_i + \kappa_i(r_i)\eta_i$$

$$\dot{\eta}_i = -\kappa_i(r_i)\eta_i + u_i$$

$$\dot{\theta}_i = \tau_i(\zeta_i, r_i)$$

$$\dot{r}_i = -\alpha\nabla f_i(r_i) - \beta \sum_{j=1}^{N} a_{ij}(r_i - r_j) - \sum_{j=1}^{N} a_{ij}(v_i - v_j)$$

$$\dot{v}_i = \alpha\beta \sum_{j=1}^{N} a_{ij}(r_i - r_j).$$

Choosing

$$\dot{e}_i \geq \hat{c}_i, \quad \dot{\Delta}_i(\hat{z}_i) \geq \hat{\phi}_2(\hat{z}_i) + 1$$

$$\rho_i(\zeta_i, r_i) \geq \hat{\gamma}_i(\zeta_i, r_i) + \hat{\phi}_1(r_i) + \hat{\phi}_2(\zeta_i) + 2$$

$$\tau_i(\zeta_i, r_i) = \rho_i(\zeta_i, r_i)\zeta_i^2, \quad \Theta_i \geq \frac{1}{2\hat{b}_0} \max\{\hat{e}_i, \hat{c}_i, 1\}$$

(20)

gives $\dot{V}_i \leq -||z_i||^2 - \zeta_i^2 + \left[\hat{e}_i, \hat{c}_i, \gamma_i(\mu_i, r_i), k_i\right] \mu_i^2$. By Lemma 2 and the smoothness of $\hat{\gamma}_i$, there exists a constant $c_{\mu_i} > 0$ satisfying $\hat{e}_i, \hat{c}_i, \gamma_i(\mu_i, r_i), k_i \leq c_{\mu_i}$, which further implies

$$\dot{V}_i \leq -||z_i||^2 - \zeta_i^2 + c_{\mu_i} \mu_i^2.$$

Step 2: we show that the closed-loop system (19) admits well-defined bounded trajectories for $t > 0$ and is globally asymptotically $e_i$-semistable at 0.

Note that the equilibria set of (19) is specified by $D = \{\operatorname{col}(z, \zeta, r, v) \mid z = 0, \zeta = 0, r = 1_N, v = v^* + 1, 1_N\}$ with an arbitrary constant $l_w$. For $e_i > 0$, we set $\Theta_i = \Theta_i(\theta_i, \ldots, \Theta_N)$ and verify that col(0, 0, $\Theta_i$, $1_N$, $v^*)$ is an equilibrium of system (19).

From the Proof of Lemma 2, we know that the function $W_i(\tau, \tau_2)$ defined thereof satisfies $\dot{\tau}_1 \leq |\tau_1| |\tau_2|^2 \leq W_0(\tau, \tau_2) \leq \tilde{\tau}_2(\tau, \tau_2)^2$, which gives some constants $\tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3 > 0$. Due to the Lipschitzness of $e_i$, $\mu_i$ is also Lipschitz in $\tau, \tau_2$. Thus, there exists a constant $\tilde{\tau}_4 > 0$ such that $\sum_{i=1}^{N} c_{\mu_i} \mu_i^2 \leq \tilde{\tau}_4^2(\tau, \tau_2)^2$. Let $V = \sum_{i=1}^{N} V_i + \tilde{\tau}_4 V$, with $V_i$ defined in the proof of Lemma 2. The first condition in Lemma 1 is verified. Taking the time derivative of $V$ along the trajectory of (19) gives

$$\dot{V} \leq -||z||^2 - \zeta^2 + \sum_{i=1}^{N} c_{\mu_i} \mu_i^2 - \tilde{\tau}_4^2(\tau, \tau_2)^2$$

This implies the second inequality in Lemma 1. Overall, the function $V$ indeed satisfies the conditions in Lemma 1. This guarantees the trajectory's boundedness over $[0, +\infty)$ and the global asymptotic $e_i$-semistability of system (19) at 0. By Lemma 3, we complete the proof.

Remark 5: The developed optimal consensus control (18) is of a high-gain type to handle the uncertainties. The parameters and functions can be sequentially constructed. First, we choose $\alpha, \beta$ according to Lemma 2. Then, we choose $\kappa_i$ according to Lemma 4. Finally, we choose $\rho_i, \tau_i$ to satisfy (20).

In some case, set $W$ or at least its boundary might be known to us. Of course, we can still use the controller (18) to solve this problem. But we can further construct a simpler controller based on the information of $W$ to this end, it is reasonable to introduce a new assumption to replace Assumption 4.

Assumption 5: For each $i \in C$, there exists a continuously differentiable function $W_i(\tau_i)$ such that, for all $r_i \in \mathbb{R}$ and $w \in \mathbb{W}$, along the trajectory of system (9)

$$\frac{\alpha_i}{\alpha_i}(|\tau_i|) \leq W_i(\tau_i) \leq \frac{\alpha_i}{\alpha_i}(|\tau_i|)$$

Choosing

$$\dot{V}_i \leq -\alpha_i(|\tau_i|) + \gamma_i(\mu_i, r_i) \mu_i^2$$

(21)

for some known smooth functions $\alpha_i, \tau_i, \alpha_i \in \mathbb{K}_{\kappa_i, \gamma_i, \gamma_i, \mu_i} > 1$ with $\alpha_i$ satisfying $\lim_{\sup_{\tau_i \rightarrow +\infty}} \alpha_i(\tau_i) < +\infty$.

In this case, we propose a reduced-order controller:

$$u_i = -\rho_i(\zeta_i, r_i)\zeta_i + \kappa_i(r_i)\eta_i$$

$$\dot{\eta}_i = -\kappa_i(r_i)\eta_i + u_i$$
\[
\dot{r}_i = -\alpha \nabla f_i(r_i) - \beta \sum_{j=1}^{N} a_{ij}(r_i - r_j) - \sum_{j=1}^{N} a_{ij}(v_i - v_j)
\]
\[
\dot{v}_i = \alpha \beta \sum_{j=1}^{N} a_{ij}(r_i - r_j).
\]

The optimal output consensus problem can be solved by this new controller as shown in the following theorem.

**Theorem 2:** Under Assumptions 1–3 and 5, there exist positive constants \(\alpha, \beta\) and positive smooth functions \(\kappa_i(r_i), \phi_i(\zeta_i, r_i)\) such that Problem 1 for multiagent system (2) is solved by a distributed controller of the form (22).

**Proof:** The proof is similar as that of Theorem 1, and we only provide some brief arguments.

First, by similar arguments as that in the proof of Lemma 4, we can show that, for each \(i \in N\), there exist a smooth function \(\kappa_i(r_i) > 0\) and a continuously differentiable function \(W_i(\hat{\zeta}_i)\) such that, along the trajectory of system (12),

\[
\dot{\hat{\zeta}}_i(\hat{\zeta}_i) \leq W_i(\hat{\zeta}_i) \leq \hat{\zeta}_i(\hat{\zeta}_i)
\]
\[
W_i \leq -\|\hat{\zeta}_i\|^2 + \gamma_i(\zeta_i, r_i)\|\zeta_i\|^2 + \gamma_i(\mu_i, r_i)\mu_i^2
\]

for some known smooth functions \(\hat{\zeta}_i, \hat{\zeta}_i \in \mathbb{K}_\infty, \gamma_i, \gamma_i > 1\).

Next, we apply the changing supply functions technique to the \(\hat{\zeta}_i\)-subsystem and conclude that, for any given smooth function \(\Delta_i(\hat{\zeta}_i) > 0\), there exists a continuously differentiable function \(W_i(\hat{\zeta}_i)\) such that, along the trajectory of (12),

\[
\dot{\hat{\zeta}}_i(\hat{\zeta}_i) \leq W_i(\hat{\zeta}_i) \leq \hat{\zeta}_i(\hat{\zeta}_i)
\]
\[
W_i \leq -\Delta_i(\hat{\zeta}_i)\|\hat{\zeta}_i\|^2 + \gamma_i(\zeta_i, r_i)\|\zeta_i\|^2 + \gamma_i(\mu_i, r_i)\mu_i^2
\]

for some known smooth functions \(\hat{\zeta}_i, \hat{\zeta}_i \in \mathbb{K}_\infty, \gamma_i, \gamma_i > 1\).

Let \(\hat{V}_i(\hat{\zeta}_i, \zeta_i) = W_i(\hat{\zeta}_i) + \zeta_i^2\). By [30, Lemma 11.1] and completing the square, one can obtain that

\[
\dot{V}_i \leq -\Delta_i(\hat{\zeta}_i)\|\hat{\zeta}_i\|^2 + \gamma_i(\mu_i, r_i)\mu_i^2
\]
\[
-2b_i(w)\rho_i(\zeta_i, r_i) - \gamma_i(\mu_i, r_i)\mu_i^2 - \phi_i(\zeta_i, r_i) - \phi_i(\zeta_i, r_i) + \phi_i(\zeta_i, r_i) - \phi_i(\zeta_i, r_i) - 1\|\zeta_i\|^2
\]

for some known smooth functions \(\phi_{i1}, \phi_{i2}, \phi_{i3} > 1\). Letting \(\Delta_i(\hat{\zeta}_i) \geq \phi_{i2}(\zeta_i) + 1, \rho_i(\zeta_i, r_i) \geq \gamma_i(\mu_i, r_i)\mu_i^2, \phi_{i1}(\zeta_i, r_i) + \phi_{i3}(\zeta_i, r_i) + 2\)

implies \(\dot{V}_i \leq -\|\hat{\zeta}_i\|^2 - \zeta_i^2 + \gamma_i(\mu_i, r_i)\mu_i^2\) for some constant \(\gamma_i > 0\). Then, the steps of Step 2 in the proof of Theorem 1 proceed as well and thus complete the proof.

Since we are supposed to know the boundary of set \(W\), no adaptive component is needed in controller (22). In this case, the rest parameters and functions and can be derived in a similar way as mentioned in Remark 5.

**Remark 6:** The controllers (18) and (22) are both composed of two parts constructed in two steps: optimal signal generator for problem (4) and distributed partial stabilizer for the augmented system composed of (7) and (9). By this two-step procedure and dynamic compensator based feedback designs, the technical difficulties brought by nonlinearity, uncertainties, and optimal requirements are successfully overcome.

**Remark 7:** Compared with [12], the multiagent system (2) is further subject to dynamic uncertainties. Moreover, the considered agents are nonlinearly parameterized with respect to uncertainties in contrast to the linear parameterized fashion in [12]. As the pure adaptive rules fail to solve this problem, a novel robust distributed controller has been developed to deal with the complicated uncertainties.

**V. SIMULATION**

In this section, we present two examples to illustrate the effectiveness of our designs.

**Example 1:** Consider a rendezvous problem [32] for four single-link manipulators with flexible joints as follows:

\[
J_1 q_{11} + M_g L_1 \sin q_{11} + k_1(q_{11} - q_{12}) = 0
\]
\[
J_2 q_{12} - k_1(q_{11} - q_{12}) = u_i
\]

where \(q_{11}, q_{12}\) are the angular positions, \(J_1, J_2\) at the moments of inertia, \(M_g\) is the total mass, \(L_1\) is a distance, \(k_1\) is a spring constant, and \(u_i\) is the torque input. The communication digraph among these agents is depicted as Fig. 1 with unity edge weights with \(k_4 = 2\) and \(k_4 = 3\).

To steer these manipulators to rendezvous at a common position that minimizes the aggregate distance from their starting position to this final position, we let \(y_1 = q_{11}\) and take the cost functions as \(f_i(y_i) = \frac{1}{2} \|y_i - q_{11}(0)\|^2 + f(y) = \frac{1}{2} \sum_{i=1}^{4} \|y - q_{11}(0)\|^2\) (1 = 1, 4, 4). One can check that the optimal solution of the global cost function is \(y^* = \frac{1}{4} \sum_{i=1}^{4} q_{11}(0)\). To make this problem more interesting, we assume that \(M_0 = (1 + w_{11})M_{10}\) and \(L_i = (1 + w_{22})L_{10}\) with nominal mass \(M_{10}\), nominal length \(L_{10}\), and uncertain parameters \(w_{11}, w_{22}\).

Letting \(x_i = \cos(q_{11}, q_{11}, q_{12}, q_{12})\), we rewrite system (24) into the form (2) with \(w = \cos(w_{11}, w_{12}, \ldots, w_{41}, w_{42})\), \(n_i = 4, b_i(w) = \frac{k_1}{4}\) and \(g_i(x_i, w) = -x_{11} \frac{M_g q_1}{L_1} \cos(x_{11}) + \frac{b_i}{4}\)

implies \(\dot{V}_i \leq -\|\hat{\zeta}_i\|^2 - \zeta_i^2 + \gamma_i(\mu_i, r_i)\mu_i^2\) for some constant \(\gamma_i > 0\). Then, the steps of Step 2 in the proof of Theorem 1 proceed as well and thus complete the proof.

Since we are supposed to know the boundary of set \(W\), no adaptive component is needed in controller (22). In this case, the rest parameters and functions and can be derived in a similar way as mentioned in Remark 5.

**Example 2:** Consider another multiagent system including two controlled FitzHugh–Nagumo dynamics [33]

\[
\dot{z}_i = - (1 + w_{13})z_i + (1 - w_{14})b_i x_i
\]
\[
\dot{x}_i = (1 + w_{16})x_i (a - x_i)(x_i - 1) - z_i + (1 + w_{16})u_i
\]
\[
y_i = x_i, \quad i = 1, 2
\]
Fig. 3. Profiles of agent outputs in Example 2.

and two controlled Van der Pol oscillators [13]

\[
\begin{align*}
\dot{x}_{12} & = (1 + w_{13})x_{12} + (1 + w_{14})(1 - x_{12}^2)x_{12} + (1 + w_{15})u_i \\
y_i & = x_{12}, \quad i = 3, 4
\end{align*}
\]

with input \(u_i\), output \(y_i\), constants \(a, b, c > 0\), and unknown parameter \(w_{ij}\). Let \(w = \text{col}(w_{13}, w_{14}, \ldots, w_{43}, w_{44})\). Clearly, all these agents are of the form (2).

We consider the optimal output consensus problem for this heterogeneous multiagent system with more complicated cost functions as

\[
\begin{align*}
f_1(y) &= (y - 8)^2, \\
f_2(y) &= \frac{y^2}{\ln(y^2 + 2)} + (y - 2)^2, \\
f_3(y) &= \ln(e^{-0.05y} + e^{0.05y}) + y^2.
\end{align*}
\]

Using the inequalities \(\frac{1}{\ln(y^2 + 2)} \leq 1.5, 0 \leq \frac{1}{\sqrt{y^2 + 1}} \leq 1, -1 \leq \frac{e^{0.05y} - e^{-0.05y}}{y} \leq 1\), we can verify Assumption 2 with \(L = 1\) and \(l = 3\) for \(i = 1, 2, \ldots, 4\). Furthermore, the global optimal point is \(y^* = 3.24\) by numerically minimizing \(\sum_{i=4} \tilde{f}_i(y)\).

Let \(a = 0.2, b = 0.8, c = 0.8\). The uncertain parameters are randomly chosen such that \(w_{13}, w_{15} \geq 0\) for \(i = 1, 2, \ldots, 4\). Without knowing the boundary of the compact set \(W\) containing these uncertainties, the controllers in [19] fail to solve the associated output consensus problem. However, we can verify Assumptions 3 and 4 for \(i = 1, 2\) with \(\gamma_{\alpha}(s, w) = (1 + w_{15})\beta \sigma_{\alpha}(s) = s^2\), \(\gamma_{\mu}(s) = \gamma_{\mu}(s) = 1\). Note that these two assumptions trivially hold for \(i = 3, 4\). According to Theorem 1, the associated output consensus problem can be solved by a distributed controller of the form (18). For simulations, we still use \(\alpha = 1, \beta = 15\), and then choose \(\rho_1(\zeta_i, r_i) = \zeta_i^2 + r_i^2 + 1, \kappa_1(r_i) = r_i^2 + 1, \tau_1(\zeta_i, r_i) = \rho_1(\zeta_i, r_i)\zeta_i^2\) with \(\zeta_i = x_{11} - r_i\) for \(i = 1, 2\) and \(\rho_2(\zeta_i, r_i) = \zeta_i^2 + r_i^2 + 1, \kappa_2(r_i) = r_i^2 + 1, \tau_2(\zeta_i, r_i) = \rho_2(\zeta_i, r_i)\zeta_i^2\) with \(\zeta_i = x_{12} - r_i\) for \(i = 3, 4\). All initial conditions are randomly chosen and the simulation result is shown in Fig. 3, where a satisfactory performance can be observed and the optimal output consensus is achieved at the optimal point \(y^* = 3.24\).

VI. CONCLUSION

We have studied an optimal output consensus problem for a class of heterogeneous high-order nonlinear systems with both static and dynamic uncertainties. We proposed a two-step design scheme to convert it into two subproblems: optimal consensus for single-integrator multiagent system and distributed partial stabilization of some augmented nonlinear systems. By adding a dynamic compensator to deal with the uncertainties, we constructed two distributed controls for this problem under standing conditions. Our future works include the MIMO extension with time-varying digraphs.

APPENDIX

PROOF OF Lemma 4

We first consider the case when \(n_1 \geq 2\). Under Assumption 4, we apply the changing supply functions technique to the \(\tau_i\)-subsystem and conclude that, for any given \(\Delta_{\tau_i}(\tau_i) > 0\), there exists a continuously differentiable function \(W_{\tau_i}^1(\tau_i)\) satisfying \(\Delta_{\tau_i}(||\tau_i||) \leq W_{\tau_i}^1(\tau_i) \leq \overline{\Delta}_{\tau_i}(||\tau_i||)\) and

\[
W_{\tau_i}^1 \leq -\Delta_{\tau_i}(\tau_i)||\tau_i||^2 + \tau \overline{\gamma}_{\tau_i}(\tau_i)||\tau_i||^2 + \sigma_{\mu\tau_i}(\mu_i, r_i)^2
\]

for some known smooth functions \(\Delta_{\tau_i}, \overline{\gamma}_{\tau_i} \in \mathcal{K}_{\infty}, \overline{\tau}_{\tau_i} > 1\) and unknown constants \(\tau, \sigma_{\mu\tau_i} > 1\).

From the choice of \(k_{ij}\), matrix \(A_i^\top\) is Hurwitz. Then, there exists a unique positive definite matrix \(P_i^0\) satisfying \(A_i^\top P_i^0 + \overline{P}_i^0 A_i^T = -3I_{n_i-1}\). Let \(W_{\xi_i}^0(\xi_i) = \chi_i P_i^0(\xi_i).\) Its time derivative along the trajectory of (9) satisfies

\[
\dot{W}_{\xi_i}^0 = 2\overline{\gamma}_{\tau_i}(A_i^\top \xi_i + B_i^\top \zeta_i - E_i^\top \mu_i) - \frac{1}{2} ||\xi_i||^2 - ||P_i^0 B_i^\top ||^2 ||\xi_i||^2 + ||P_i^0 E_i^\top ||^2 ||\mu_i||^2.
\]

By changing supply functions of \(\xi_i\)-subsystem, for any given \(\Delta_{\tau_i}(\tau_i) > 0\), there exists a continuously differentiable function \(W_{\tau_i}^1(\tau_i)\) satisfying \(\Delta_{\tau_i}(||\tau_i||) \leq W_{\tau_i}^1(\tau_i) \leq \overline{\Delta}_{\tau_i}(||\tau_i||)\) and

\[
W_{\tau_i}^1 \leq -\Delta_{\tau_i}(\tau_i)||\tau_i||^2 + \tau \overline{\gamma}_{\tau_i}(\tau_i)||\tau_i||^2 + \sigma_{\mu\tau_i}(\mu_i, r_i)^2
\]

for some known smooth functions \(\Delta_{\tau_i}, \overline{\gamma}_{\tau_i} \in \mathcal{K}_{\infty}, \overline{\tau}_{\tau_i} > 1\). Clearly, there exist functions \(\tilde{\alpha}_{\tau_i}, \tilde{\gamma}_{\tau_i} \in \mathcal{K}_{\infty}\) satisfying \(\tilde{\alpha}_{\tau_i}(||\tau_i||) \leq W_{\tau_i}^1(\tau_i) \leq \tilde{\gamma}_{\tau_i}(||\tau_i||)\) and its time derivative along the trajectory of (12) satisfies

\[
\dot{W}_{\tau_i}^1 \leq -\tilde{\alpha}_{\tau_i}(\tau_i)||\tau_i||^2 - \tilde{\gamma}_{\tau_i}(\tau_i)||\tau_i||^2 + \sigma_{\mu\tau_i}(\mu_i, r_i)^2.
\]

When \(n_i = 1\), the above property trivially holds for \(\bar{z}_i = \bar{z}_i\).

Next, we apply the changing supply functions technique to \(\bar{z}_i\)-subsystem and conclude that, for any given smooth function \(\Delta_{\bar{z}_i}(\bar{z}_i) > 0\), there exists a continuously differentiable function \(W_{\bar{z}_i}^1(\bar{z}_i)\) satisfying \(\Delta_{\bar{z}_i}(||\bar{z}_i||) \leq W_{\bar{z}_i}^1(\bar{z}_i) \leq \overline{\Delta}_{\bar{z}_i}(||\bar{z}_i||)\) and

\[
W_{\bar{z}_i}^1 \leq -\Delta_{\bar{z}_i}(\bar{z}_i)||\bar{z}_i||^2 + \tilde{\sigma}_{\mu\bar{z}_i}(\bar{z}_i)||\bar{z}_i||^2 + \sigma_{\mu\bar{z}_i}(\mu_i, r_i)^2
\]

for some known smooth functions \(\Delta_{\bar{z}_i}, \tilde{\sigma}_{\mu\bar{z}_i} \in \mathcal{K}_{\infty}, \tilde{\tau}_{\bar{z}_i}, \tilde{\tau}_{\bar{z}_i} > 1\), and unknown constants \(\tilde{\sigma}_{\mu\bar{z}_i} > 1\).

Let \(W_{\bar{z}_i}(\bar{z}_i) = \bar{\xi}_i W_{\bar{z}_i}^1(\bar{z}_i) + \bar{\eta}_i\) with \(\bar{\xi}_i > 0\) to be specified later. Clearly, the first inequality in Lemma 4 holds. We take time derivative of \(W_i\) along the trajectory of (19) and have

\[
\dot{W}_i \leq -\bar{\xi}_i \Delta_{\bar{z}_i}(\bar{z}_i)||\bar{z}_i||^2 - \sigma_{\mu\bar{z}_i}(\bar{z}_i)||\bar{z}_i||^2 + \tilde{\sigma}_{\mu\bar{z}_i}(\mu_i, r_i)^2\]

\[
+ \{ -\kappa_i(r_i)\bar{\xi}_i + \bar{\zeta}_i(\bar{z}_i, \xi_i, \zeta_i, r_i, w) + \psi_i(r_i, w)\mu_i\}.
\]
Jointly with the inequalities (13) and (14), we can bound the cross terms by completing the square and have

\[ \dot{W}_i \leq \left[ \hat{\theta}_i \mathbf{\Delta}_i (\hat{z}_i) - 2 \hat{\kappa}_i \hat{\hat{\phi}}_i (\hat{z}_i) \right] ||\hat{z}_i||^2 \]

\[ - \kappa_i (r_i - \frac{\hat{\phi}_i (r_i)}{2}) - \frac{1}{2} \hat{\eta}_3^2 \]

\[ + \left[ \hat{\hat{\phi}}_i \mathbf{\sigma}_i (\hat{z}_i) \gamma_i (r_i, m_i, r_i) \right] \mu_i^2 \]

Note that \( \kappa_i (r_i) \geq \hat{\phi}_i (r_i) + 1 \). Letting \( \hat{\theta}_i > \frac{2 \hat{\phi}_i}{b_0} + 1, \hat{\Delta}_i (\hat{z}_i) \geq \hat{\phi}_i (\hat{z}_i) + 1, \hat{\sigma}_i > \hat{\hat{\phi}}_i \mathbf{\sigma}_i + \frac{2 \hat{\phi}_i}{b_0} \hat{\sigma}_i + \hat{\hat{\phi}}_i \mathbf{\sigma}_i + 2 \hat{\phi}_i \mathbf{\sigma}_i \), and \( \hat{\phi}_i (r_i, m_i, r_i) > \hat{\phi}_i (r_i) + \hat{\phi}_i (r_i) \) implies the second inequality and thus completes the proof.

**References**


