IEEE CONTROL SYSTEMS LETTERS, VOL. 5, NO. 4, OCTOBER 2021

Multi-Agent Optimal Consensus With Unknown Control Directions

Yutao Tang, *Member, IEEE*

***Abstract*—This letter studies the optimal consensus problem for a group of heterogeneous high-order agents with unknown control directions. Compared with existing consensus results, reaching an optimal consensus further requires the consensus point to be a solution to some distributed optimization problem. First, we construct an optimal signal generator to reproduce the global optimal point. Then, we convert the optimal consensus problem into a robust stabilization problem for some translated multi-agent system. Finally, we present two distributed adaptive controllers depending upon different available information. Numerical examples are given to verify the efficacy of our designs.**

***Index Terms*—Optimal consensus, high-order dynamics, unknown control direction, adaptive control.**

I. Introduction

CONSENSUS problems have been extensively studied for decades in multi-agent coordination literature. Particularly, distributed consensus optimization or optimal consensus attracts much attention and has many potential applications in multi-robot networks and large-scale machine learning problems. In distributed consensus optimization, each agent has a local cost function and all agents are expected to reach a consensus state that minimizes the sum of their individual cost functions. Many effective algorithms have been proposed for single-integrator multi-agent systems under various conditions (see [12], [20], [29] and references therein).

Along with these results for single integrators, there are numerous optimal consensus tasks implemented by or depending on engineering multi-agent systems of high-order dynamics, e.g., source seeking in mobile sensor networks [31], frequency control in power systems [32], and attitude formation control of rigid bodies [21]. Thus, many authors seek to solve the optimal consensus problem for non-single-integrator multi-agent systems. Some recent attempts include [28], [33] for second-order agents, [23] for general linear agents, and [25]–[27] for several special classes of nonlinear agents.

Manuscript received May 17, 2020; revised July 25, 2020; accepted August 18, 2020. Date of publication September 3, 2020; date of current version September 18, 2020. This work was supported by the National Natural Science Foundation of China under Grant 61973043. Recommended by Senior Editor L. Menini.

The author is with the School of Artificial Intelligence, Beijing University of Posts and Telecommunications, Beijing 100876, China (e-mail: yttang@bupt.edu.cn).

Digital Object Identifier 10.1109/LCSYS.2020.3021635

So far, all these optimal consensus results were derived assuming the knowledge of the control directions of agents' dynamics. However, this information may not always be available beforehand. Even it is known at first, it could be changed by some structural damages in many applications as shown in [4], [11]. Therefore, it is crucial to consider the optimal consensus problem for engineering multi-agent systems when the control directions are unknown.

At the same time, plenty of consensus results without such optimization requirements have been derived for multi-agent systems with unknown control directions by extending the classical Nussbaum-type controls [13], [30] to decentralized and distributed cases, e.g., [1], [6], [14], [18]. It is thus very interesting to ask whether similar Nussbaum-type controls can be constructed to tackle the optimal consensus problem in the presence of unknown control directions.

Based on the aforementioned observations, we consider a group of high-order multi-agent systems with unknown control directions and seek distributed gradient-based rules to solve the associated optimal consensus problem. Although some interesting optimal consensus results have been partially obtained for these agents [23], [28], [33], their optimal consensus problem in the presence of unknown control directions is more challenging and its solvability is still unclear. In fact, gradient-based rules are basically nonlinear in light of the optimization requirement. More importantly, the unknown control directions and heterogeneous system orders of these agents bring many extra technical difficulties to the associated optimal consensus analysis and design.

To solve the formulated optimal consensus problem, we are going to develop an embedded control-based rule as that in [23]. We will first assume the local cost function's analytic form and present an optimal signal generator to reproduce the global optimizer in a distributed manner. Then, the expected optimal consensus can be achieved by solving a robust stabilization problem for some translated multi-agent system. After that, we will present an extension when only real-time numerical gradients of the local cost functions are available.

The main contribution of this letter can be summarized as follows. First, compared with existing optimal consensus results requiring a prior knowledge of agents' control directions [15], [23], [28], we remove this limitation and present effective distributed controllers for these heterogeneous high-order agents to reach an optimal consensus in the presence of unknown control directions. To our knowledge, no other work solves such an optimal consensus problem under these circumstances yet. Second, as (average) consensus can be achieved by solving some special optimal consensus problem, our algorithms naturally provide an alternative way other than [1],

[7], [14], [18] to tackle such (average) consensus problems for agents with unknown control directions extending the consensus results derived in [16], [17].

This letter is organized as follows. Some preliminaries are listed in Section II with problem formulation in Section III. Main results are presented in Section IV. Finally, simulations and conclusions are given in Sections V and VI.

II. Preliminaries

We will use standard notations. Let $\mathbb{R}^N$ represent the N-dimensional Euclidean space. Denote by $\|a\|$ the Euclidean norm of a vector a and by $\|A\|$ the spectral norm of a matrix A. $\mathbf{1}_N$ (or $\mathbf{0}_N$) denotes the N-dimensional all-one (or all-zero) column vector, and I_N denotes the N-dimensional identity matrix. Set $M_1 = \frac{1}{\sqrt{N}}\mathbf{1}_N$ and let M_2 be the matrix satisfying $M_2^\top M_1 = \mathbf{0}_{N-1}$, $M_2^\top M_2 = I_{N-1}$ and $M_2 M_2^\top = I_N - M_1 M_1^\top$. We may omit the subscript when it is self-evident.

A directed graph (digraph) is described by $\mathscr{G} = (\mathscr{N}, \mathscr{E}, \mathscr{A})$ with node set $\mathscr{N} = \{1, \ldots, N\}$ and edge set $\mathscr{E}$. $(i, j) \in \mathscr{E}$ denotes an edge from node i to j. The weighted adjacency matrix $\mathscr{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is defined by $a_{ii} = 0$ and $a_{ij} \geq 0$. Here $a_{ij} > 0$ iff $(j, i) \in \mathscr{E}$. Node i's neighbor set is defined as $\mathscr{N}_i = \{j \mid (j, i) \in \mathscr{E}\}$. A directed path is an ordered sequence of nodes such that each intermediate pair of nodes is an edge. A digraph is said to be strongly connected if there is a directed path between any two nodes. The in-degree and out-degree of node i are defined as $d_i^{\text{in}} = \sum_{j=1}^N a_{ij}$ and $d_i^{\text{out}} = \sum_{j=1}^N a_{ji}$. A digraph is weight-balanced if $d_i^{\text{in}} = d_i^{\text{out}}$ for any $i \in \mathscr{N}$. The Laplacian of digraph $\mathscr{G}$ is defined as $L \triangleq D^{\text{in}} - \mathscr{A}$ with $D^{\text{in}} = \text{diag}(d_1^{\text{in}}, \ldots, d_N^{\text{in}})$. Note that $L\mathbf{1}_N = \mathbf{0}_N$ for any digraph. A digraph is weight-balanced iff $\mathbf{1}_N^\top L = \mathbf{0}_N$, which is also equivalent to $\text{Sym}(L) = \frac{L+L^\top}{2}$ being positive semidefinite. For a strongly connected and weight-balanced digraph, we can order the eigenvalues of $\text{Sym}(L)$ as $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$ and have $\lambda_2 I_{N-1} \leq M_2^\top \text{Sym}(L) M_2 \leq \lambda_N I_{N-1}$. See [5] for details.

A function $f : \mathbb{R}^m \to \mathbb{R}$ is said to be convex if $f(a\zeta_1 + (1-a)\zeta_2) \leq af(\zeta_1) + (1-a)f(\zeta_2)$ for any $0 \leq a \leq 1$ and all $\zeta_1, \zeta_2 \in \mathbb{R}^m$. When f is differentiable, it is convex if $f(\zeta_1) - f(\zeta_2) \geq \nabla f(\zeta_2)^\top(\zeta_1 - \zeta_2)$ for all $\zeta_1, \zeta_2 \in \mathbb{R}^m$. We say f is ω-strongly convex over $\mathbb{R}^m$ if $[\nabla f(\zeta_1) - \nabla f(\zeta_2)]^\top(\zeta_1 - \zeta_2) \geq \omega\|\zeta_1 - \zeta_2\|^2$ for all $\zeta_1, \zeta_2 \in \mathbb{R}^m$ with $\omega > 0$. A vector-valued function $\boldsymbol{f} : \mathbb{R}^m \to \mathbb{R}^m$ is said to be ϑ-Lipschitz if $\|\boldsymbol{f}(\zeta_1) - \boldsymbol{f}(\zeta_2)\| \leq \vartheta\|\zeta_1 - \zeta_2\|$ for all $\zeta_1, \zeta_2 \in \mathbb{R}^m$ with $\vartheta > 0$.

III. Problem Formulation

Consider a heterogeneous multi-agent system consisting of N agents described by

$$y_i^{(n_i)} = b_i u_i, \quad i \in \mathscr{N} \triangleq \{1, \ldots, N\} \tag{1}$$

where $y_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are its output and input, respectively. Integer $n_i \geq 1$ is the order of system (1) and constant b_i is assumed to be nonzero but unknown. This constant b_i is often called the high-frequency gain of agent (1), and it represents the motion direction of this agent in any control strategy. The parameters n_i and b_i of each agent are allowed to be different from each other.

We endow agent i with a local cost function $f_i : \mathbb{R} \to \mathbb{R}$ for $i \in \mathscr{N}$ and define the global cost function as $f(y) = \sum_{i=1}^N f_i(y)$. For multi-agent system (1), we aim to develop an algorithm such that all agent outputs achieve a consensus on a minimizer of this global cost function f.

Assumption 1: There exist two constants $\bar{l}$, $\underline{l} > 0$ such that f_i is $\underline{l}$-strongly convex and its gradient is $\bar{l}$-Lipschitz for $i \in \mathscr{N}$.

This assumption has been widely used in [8], [10], [19], [26], and it guarantees the existence and uniqueness of the minimal solution to function f. As usual, we assume this optimal solution is finite and denote it as $y^\star$, i.e.,

$$y^\star = \arg\min_{y\in\mathbb{R}} f(y) = \sum_{i=1}^N f_i(y) \tag{2}$$

Due to the privacy of local cost function f_i, no agent can unilaterally determine the global minimizer $y^\star$ by itself. Hence, our problem cannot be solved without information sharing among agents. We use a digraph $\mathscr{G} = (\mathscr{N}, \mathscr{E}, \mathscr{A})$ to describe the information sharing topology, where edge $(j, i) \in \mathscr{E}$ means that agent i can get the information of agent j.

Assumption 2: Digraph $\mathscr{G}$ is weight-balanced and strongly connected.

Then, our optimal consensus problem is to design u_i for agent i under the information constraint described by digraph $\mathscr{G}$, such that these agents achieve an optimal consensus determined by the global cost function f in the sense that $y_i - y^\star \to 0$ as $t \to \infty$ for any $i \in \mathscr{N}$, while the trajectories of this multi-agent system are maintained to be bounded.

Remark 1: This optimal consensus problem has been extensively studied in literature for multi-agent systems assuming that the high-frequency gain is known [15], [23], [28], [33]. But in this letter, this prior knowledge of each agent's control direction is no longer necessary, which means that agents may have different and unknown control directions. To the best of our knowledge, no other works have studied the optimal consensus problem under these assumptions yet.

Note that when the local cost functions are specified as $f_i(y) = c_i(y - y_i(0))^2$ with $c_i > 0$ for each $i \in \mathscr{N}$, all agents achieve a scaled consensus with the consensus point as $y^\star = \frac{\sum_{i=1}^N c_i y_i(0)}{\sum_{i=1}^N c_i}$. Thus, this formulation provides an applicable way to solve their average consensus problems for these high-order agents in the presence of unknown control directions.

IV. Main Result

In this section, we will present an embedded control-based design to solve our formulated optimal consensus problem following the technical line developed in [23].

To this end, we first consider an auxiliary optimal consensus problem with the same requirement for agents in the form $\dot{r}_i = \mu_i$ and then tackle our original optimal consensus problem by solving an output tracking problem for agent (1) with reference r_i. As the former subproblem is essentially a conventional optimal consensus problem for single integrators with $b_i = 1$ and has been well-studied in existing literature, we use the following optimal signal generator to complete our design:

$$\begin{aligned}
\dot{r}_i &= -\alpha\nabla f_i(r_i) - \beta\sum_{j=1}^N a_{ij}(r_i - r_j) - \sum_{j=1}^N a_{ij}(v_i - v_j)\\
\dot{v}_i &= \alpha\beta\sum_{j=1}^N a_{ij}(r_i - r_j)
\end{aligned} \tag{3}$$

where α, β are constants to be specified later. System (3) is a distributed primal-dual variant to determine the optimal consensus point $y^\star$. Its effectiveness has already been established in [26] by semistability arguments. Here, we provide a sketch of proof using Lyapunov stability analysis.

Lemma 1: Suppose Assumptions 1–2 hold and let $\alpha \geq \max\{1, \frac{1}{\underline{l}}, \frac{2\bar{l}^2}{\underline{l}\lambda_2}\}$, $\beta \geq \max\{1, \frac{1}{\lambda_2}, \frac{6\alpha^2\lambda_N^2}{\lambda_2^2}\}$. Then, the trajectory of system (3) from any initial point is bounded over $[0, \infty)$ and $r_i(t)$ approaches $y^\star$ exponentially as $t \to \infty$ for $i \in \mathcal{N}$.

Proof: Putting system (3) into a compact form gives

$$\dot{r} = -\alpha\nabla\tilde{f}(r) - \beta Lr - Lv, \quad \dot{v} = \alpha\beta Lr \tag{4}$$

where $r = \mathrm{col}(r_1, \ldots, r_N)$, $v = \mathrm{col}(v_1, \ldots, v_N)$, and $\tilde{f}(r) \triangleq \sum_{i=1}^N f_i(r_i)$ is $\underline{l}$-strongly convex and its gradient $\nabla\tilde{f}(r)$ is $\bar{l}$-Lipschitz in r under Assumption 1. Let $\mathrm{col}(r^\star, v^\star)$ be any equilibrium point of system (4). It can be verified that $r^\star = \mathbf{1}_N y^\star$ under Assumptions 1–2 by [19, Th. 3.27].

Performing the coordinate transformation: $\bar{r}_1 = M_1^\top(r - r^\star)$, $\bar{r}_2 = M_2^\top(r - r^\star)$, $\bar{v}_1 = M_1^\top(v - v^\star)$, and $\bar{v}_2 = M_2^\top[(v + \alpha r) - (v^\star + \alpha r^\star)]$ gives $\dot{\bar{v}}_1 = 0$ and

$$\begin{aligned}
\dot{\bar{r}}_1 &= -\alpha M_1^\top \mathbf{\Pi}\\
\dot{\bar{r}}_2 &= -\alpha M_2^\top \mathbf{\Pi} - \beta M_L\bar{r}_2 + \alpha M_L\bar{r}_2 - M_L\bar{v}_2\\
\dot{\bar{v}}_2 &= -\alpha M_L\bar{v}_2 + \alpha^2 M_L\bar{r}_2 - \alpha^2 M_2^\top \mathbf{\Pi}
\end{aligned}\tag{5}$$

where $\mathbf{\Pi} \triangleq \nabla\tilde{f}(r) - \nabla\tilde{f}(r^\star)$ and $M_L = M_2^\top L M_2$. Note that this reduced-order system has a unique equilibrium at the origin.

Let $\bar{r} = \mathrm{col}(\bar{r}_1, \bar{r}_2)$ and choose a Lyapunov function candidate for the reduced-order system (5) as $W_{\mathrm{o}}(\bar{r}, \bar{v}_2) = \bar{r}^\top\bar{r} + \frac{1}{\alpha^3}\bar{v}_2^\top\bar{v}_2$. W_{o} is quadratic and positive definite. Under the lemma condition, we take the time derivative of $W_{\mathrm{o}}(t)$ along the trajectory of (5) and obtain that

$$\dot{W}_{\mathrm{o}} \leq -\frac{1}{2}W_{\mathrm{o}}$$

where we use Young's inequality to handle the possible cross terms as that in [26]. According to [9, Th. 4.10], $W_{\mathrm{o}}(\bar{r}(t), \bar{v}_2(t))$ and $\bar{r}(t)$ will exponentially converge to 0 as t goes to infinity. Since $\dot{\bar{v}}_1 = 0$, we also confirm the boundedness of all trajectories of system (4) from any initial point over the time interval $[0, \infty)$. Moreover, by using the fact that $r - r^\star = M_1\bar{r}_1 + M_2\bar{r}_2$, one can conclude the exponential convergence of $r_i(t)$ to $y^\star$ as $t \to \infty$. ■

With this generator (3), each agent can get an asymptotic estimate of the global optimizer $y^\star$. Thus, we are left to solve the output tracking problem for agent i with reference r_i.

When $b_i = 1$, a pole-placement based tracking controller was presented in [23] for agent (1) to complete the whole design. Controllers with bounded constraints were also developed to achieve an optimal consensus in [15], [28]. However, the control directions are assumed to be unknown in our current case. Consequently, such methods are no longer applicable to agent (1) and we have to seek new tracking rules to solve our optimal consensus problem.

For this purpose, we assume $y_{i1} = y_i - r_i$ and $y_{i\iota} \triangleq \varepsilon^{\iota-1}y_i^{(\iota-1)}$ for $2 \leq \iota \leq n_i$ with a constant $\varepsilon > 0$ to be specified later. Choose constants $k_{i\iota}$ for $1 \leq \iota \leq n_i - 1$ such that the polynomial $p_i(\lambda) = \sum_{\iota=1}^{n_i-1} k_{i\iota}\lambda^{\iota-1} + \lambda^{n_i-1}$ is Hurwitz. Letting $z_i = \mathrm{col}(y_{i1}, \ldots, y_{in_i-1})$ and $\zeta_i = \sum_{\iota=1}^{n_i-1} k_{i\iota}y_{i\iota} + y_{in_i}$ gives the following translated multi-agent system:

$$\dot{z}_i = \frac{1}{\varepsilon}A_{i1}z_i + \frac{1}{\varepsilon}A_{i2}\zeta_i + E_{i1}\dot{r}_i$$

$$\dot{\zeta}_i = \frac{1}{\varepsilon}A_{i3}z_i + \frac{1}{\varepsilon}A_{i4}\zeta_i + \varepsilon^{n_i-1}b_iu_i + E_{i2}\dot{r}_i \tag{6}$$

where the associated matrices are defined as follows.

$$A_{i1} = \left[\begin{array}{c|ccc} \mathbf{0}_{n_i-2} & & I_{n_i-2} & \\ \hline -k_{i1} & -k_{i2} & \cdots & -k_{i\, n_i-1}\end{array}\right], \quad A_{i2} = \begin{bmatrix}\mathbf{0}_{n_i-2}\\ 1\end{bmatrix}$$

$$A_{i3} = \begin{bmatrix}-k_{in_i-1}k_{i1} & k_{i1} - k_{in_i-1}k_{i2} & \cdots & k_{in_i-2} - k_{in_i-1}^2\end{bmatrix}$$

$$A_{i4} = k_{in_i-1}, \quad E_{i1} = \begin{bmatrix}1 & \mathbf{0}_{n_i-2}^\top\end{bmatrix}^\top, \quad E_{i2} = -k_{i1}$$

From the proof of Lemma 1, system (5) is exponentially stable at the origin. Recalling the fact that $\dot{r} = M_1\dot{\bar{r}}_1 + M_2\dot{\bar{r}}_2$, one can conclude the exponential convergence of $\dot{r}(t)$ to $\mathbf{0}$ by using the Lipschitzness of Π under Assumption 1. Thus, we only have to seek a robust stabilizer for the translated system (6) with time-decaying disturbances $\dot{r}_i$.

Inspired by the adaptive controllers used in [2], [14], [30], we propose the following Nussbaum-type rule for (6):

$$u_i\, \overline{\mathcal{N}}(\theta_i)\zeta_i, \quad \dot{\theta}_i = \zeta_i^2$$

where $\overline{\mathcal{N}}$ is a smooth function satisfying:

$$\limsup_{\theta\to\infty} \frac{\int_0^\theta \overline{\mathcal{N}}(s)\mathrm{d}s}{\theta} = \infty, \quad \liminf_{\theta\to\infty} \frac{\int_0^\theta \overline{\mathcal{N}}(s)\mathrm{d}s}{\theta} = -\infty \tag{7}$$

Commonly used examples include $\theta^2\sin\theta$ and $e^{\theta^2}\sin\theta$.

The overall optimal consensus controller for agent i is then:

$$\begin{aligned}
u_i &= \overline{\mathcal{N}}(\theta_i)\zeta_i\\
\dot{\theta}_i &= \zeta_i^2\\
\dot{r}_i &= -\alpha\nabla f_i(r_i) - \beta\sum_{j=1}^N a_{ij}(r_i - r_j) - \sum_{j=1}^N a_{ij}(v_i - v_j)\\
\dot{v}_i &= \alpha\beta\sum_{j=1}^N a_{ij}(r_i - r_j)
\end{aligned}\tag{8}$$

where $\zeta_i = k_{i1}(y_i - r_i) + \sum_{\iota=2}^{n_i-1} k_{i\iota}\varepsilon^{\iota-1}y_i^{(\iota-1)} + \varepsilon^{n_i-1}y_i^{(n_i-1)}$ defined as above. This controller is indeed distributed in the sense of using only agent i's own and neighboring information.

It is time to present our first main theorem of this letter.

Theorem 1: Consider the multi-agent system consisting of N agents given by (1). Suppose Assumptions 1–2 hold. Then, there exist two positive constants α and β such that the optimal consensus problem for this multi-agent system (1) and (2) is solved by the controller (8) for any $\varepsilon > 0$.

Proof: According to Lemma 1 and the above problem conversion arguments, we only have to show that the trajectory of the translated system (6) from any initial point is well-defined over the time interval $[0, \infty)$ and y_{i1} converges to zero.

To this end, we first show that the trajectory of this multi-agent system is well-defined over the time interval $[0, \infty)$. Note that the local error system for agent i is

$$\begin{aligned}
\dot{z}_i &= \frac{1}{\varepsilon}A_{i1}z_i + \frac{1}{\varepsilon}A_{i2}\zeta_i + E_{i1}\dot{r}_i\\
\dot{\zeta}_i &= \frac{1}{\varepsilon}A_{i3}z_i + \frac{1}{\varepsilon}A_{i4}\zeta_i + \varepsilon^{n_i-1}b_i\overline{\mathcal{N}}(\theta_i)\zeta_i + E_{i2}\dot{r}_i\\
\dot{\theta}_i &= \zeta_i^2\\
\dot{r}_i &= -\alpha\nabla f_i(r_i) - \beta\sum_{j=1}^N a_{ij}(r_i - r_j) - \sum_{j=1}^N a_{ij}(v_i - v_j)
\end{aligned}$$

$$\dot{v}_i = \alpha\beta \sum_{j=1}^{N} a_{ij}(r_i - r_j)$$

where A_{i1} is Hurwitz due to the choice of k_{i_L}. Thus, there must be a positive definite matrix $P_i \in \mathbb{R}^{(n_i-1)\times(n_i-1)}$ such that $A_{i1}^\top P_i + P_i A_{i1} = -2I_{n_i-1}$. Since the righthand side of the above system is smooth, the trajectory of each subsystem must be well-defined on its maximal interval $[0,\, t_{if})$. We claim that $t_{if} = \infty$ for each $i \in \mathcal{N}$.

Take $V_i(z_i,\, \zeta_i) = z_i^\top P_i z_i + \frac{1}{2}\zeta_i^2$ as a sub-Lyapunov function for agent i. It is positive definite with a time derivative along the trajectory of the above error system as follows.

$$\begin{aligned}
\dot{V}_i &= 2z_i^\top P_i[\frac{1}{\varepsilon}A_{i1}z_i + \frac{1}{\varepsilon}A_{i2}\zeta_i + E_{i1}\dot{r}_i] \\
&\quad + \zeta_i(\frac{1}{\varepsilon}A_{i3}z_i + \frac{1}{\varepsilon}A_{i4}\zeta_i + \varepsilon^{n_i-1}b_i\overline{\mathcal{N}}(\theta_i)\zeta_i + E_{i2}\dot{r}_i) \\
&\le -\frac{2}{\varepsilon}\|z_i\|^2 + \frac{1}{3\varepsilon}\|z_i\|^2 + \frac{3}{\varepsilon}\|P_iA_{i2}\|^2\zeta_i^2 + \frac{1}{3\varepsilon}\|z_i\|^2 \\
&\quad + 3\varepsilon\|P_iE_{i1}\|^2\dot{r}_i^2 + \frac{1}{3\varepsilon}\|z_i\|^2 + \frac{3}{\varepsilon}\|A_{i3}\|^2\zeta_i^2 + \frac{1}{\varepsilon}|A_{i4}|\zeta_i^2 \\
&\quad + \varepsilon^{n_i-1}b_i\overline{\mathcal{N}}(\theta_i)\zeta_i^2 + \frac{1}{\varepsilon}\zeta_i^2 + \varepsilon\|E_{i2}\|^2\dot{r}_i^2 \\
&= -\frac{1}{\varepsilon}\|z_i\|^2 + (\varepsilon^{n_i-1}b_i\overline{\mathcal{N}}(\theta_i) + C_{i\theta_1})\zeta_i^2 + C_{i\theta_2}\dot{r}_i^2 \qquad (9)
\end{aligned}$$

where we use Young's inequality to handle the cross terms with constants $C_{i\theta_1} = \frac{1}{\varepsilon}(3\|P_iA_{i2}\|^2 + 3\|A_{i3}\|^2 + |A_{i4}| + 1)$ and $C_{i\theta_2} = 3\varepsilon\|P_iE_{i1}\|^2 + \varepsilon\|E_{i2}\|^2$.

Recalling Lemma 1, $r_i(t)$ and $\dot{r}_i(t)$ exponentially converge to $y^\star$ and 0 under Assumptions 1–2. Thus, $\dot{r}_i(t)$ is square-integrable over $[0,\, \infty)$. Denote $V_i(t) \triangleq V_i(z_i(t),\, \zeta_i(t))$ for short. Noting that $\dot{\theta}_i = \zeta_i^2$, we integrate both sides of (9) from 0 to t and have the following inequality for some constant $C_{i0} > 0$:

$$V_i(t) - V_i(0) \le \varepsilon^{n_i-1}b_i\int_{\theta_i(0)}^{\theta_i(t)}\overline{\mathcal{N}}(s)\,\mathrm{d}s + C_{i\theta_1}\theta_i(t) + C_{i0}$$

As $\theta_i(t)$ is monotonically increasing, it either has a finite limit or grows to ∞. Assuming $\theta_i(t)$ tends to ∞, we divide both sides by $\theta_i(t)$ for a large enough t and have

$$0 \le \varepsilon^{n_i-1}b_i\frac{\int_{\theta_i(0)}^{\theta_i(t)}\overline{\mathcal{N}}(s)\,\mathrm{d}s}{\theta_i(t)} + C_{i\theta_1} + \frac{C_{i0} + V_i(0)}{\theta_i(t)}$$

Due to the property (7), this inequality will finally be violated for any fixed b_i. Hence, $\theta_i(t)$ must be bounded over $[0,\, t_{if})$. Recalling the controller (8), $z_i(t)$, $\zeta_i(t)$, $u_i(t)$, $\dot{\zeta}_i(t)$, and $\dot{\theta}_i(t)$ are also bounded over $[0,\, t_{if})$. That is, no finite-time escape phenomenon happens. Therefore, we have $t_{if} = \infty$.

From the boundedness of $\dot{\theta}_i$, the function $\theta_i(t)$ is uniformly continuous with respect to time t. Note that

$$\int_0^t \zeta_i^2(s)\mathrm{d}s = \int_0^t \dot{\theta}_i(s)\mathrm{d}s \le \theta_i(\infty) - \theta_i(0)$$

Since $\theta_i(\infty)$ exists and is finite, $\zeta_i^2(t)$ is thus integrable. By [9, Lemma 8.2], we have $\zeta_i(t) \to 0$ as t goes to ∞.

Considering the z_i-subsystem, it is input-state stable with input $\frac{1}{\varepsilon}A_{i2}\zeta_i + E_{i1}\dot{r}_i$ and state z_i. Since both $\zeta_i(t)$ and $\dot{r}_i(t)$ converge to 0 when t goes to ∞, we use [22, Th.1] and obtain that $y_{i1}(t) = y_i(t) - r_i(t) \to 0$ as t goes to ∞. Recalling the exponential convergence of $r_i(t)$ to $y^\star$, we further conclude that $|y_i(t) - y^\star| \le |y_i(t) - r_i(t)| + |r_i(t) - y^\star| \to 0$ as t goes to ∞. The proof is thus complete. ■

In controller (8), we assume the analytic form of ∇f_i to ensure the feasibility of our optimal signal generator (3). However, in many cases, only real-time gradient $\nabla f_i(y_i)$ is available for agent i and the controller (8) is thus not implementable.

To tackle this issue, we limit us to the case when all high-frequency gains have the same sign. Replacing $\nabla f_i(r_i)$ with the real-time gradient $\nabla f_i(y_i)$, we present the following controller:

$$\begin{aligned}
u_i &= \overline{\mathcal{N}}(\theta_i)\zeta_i \\
\dot{\theta}_i &= \zeta_i^2 \\
\dot{r}_i &= -\alpha\nabla f_i(y_i) - \beta\sum_{j=1}^{N}a_{ij}(r_i - r_j) - \sum_{j=1}^{N}a_{ij}(v_i - v_j) \\
\dot{v}_i &= \alpha\beta\sum_{j=1}^{N}a_{ij}(r_i - r_j) \qquad (10)
\end{aligned}$$

where ζ_i is defined as in (8) and $\overline{\mathcal{N}}$ is strengthened to satisfy

$$\begin{aligned}
&\lim_{\theta\to\infty}\frac{\int_0^y\overline{\mathcal{N}}_+(s)\mathrm{d}s}{\theta} = \infty, \quad \limsup_{\theta\to\infty}\frac{\int_0^y\overline{\mathcal{N}}_+(s)\mathrm{d}s}{\int_0^y\overline{\mathcal{N}}_-(s)\mathrm{d}s} = \infty \\
&\lim_{\theta\to\infty}\frac{\int_0^y\overline{\mathcal{N}}_-(s)\mathrm{d}s}{\theta} = \infty, \quad \limsup_{\theta\to\infty}\frac{\int_0^y\overline{\mathcal{N}}_-(s)\mathrm{d}s}{\int_0^y\overline{\mathcal{N}}_+(s)\mathrm{d}s} = \infty \qquad (11)
\end{aligned}$$

with $\overline{\mathcal{N}}_+(\theta) \triangleq \max\{\overline{\mathcal{N}}(\theta),\, 0\}$, $\overline{\mathcal{N}}_-(\theta) \triangleq \max\{-\overline{\mathcal{N}}(\theta),\, 0\}$ for any $\theta \in \mathbb{R}$. It can be verified that such functions satisfy the condition (7) and thus are special Nussbaum functions. Some feasible examples have been used in [2], [3].

Theorem 2: Consider the multi-agent system consisting of N agents given by (1). Suppose all high-frequency gains are unknown but with the same sign and Assumptions 1–2 hold. Then, there exist positive constants α, β, and $\varepsilon^\star$ such that the optimal consensus problem for this multi-agent system (1) and (2) is solved by the controller (10) for any $0 < \varepsilon < \varepsilon^\star$.

Proof: Fixing α, β as in Lemma 1, we will decrease ε to compensate the discrepancy between $\nabla f_i(r_i)$ and $\nabla f_i(y_i)$.

By the proof of Lemma 1, the composite system in this case can be written as follows:

$$\begin{aligned}
\dot{z}_i &= \frac{1}{\varepsilon}A_{i1}z_i + \frac{1}{\varepsilon}A_{i2}\zeta_i + E_{i1}\dot{r}_i \\
\dot{\zeta}_i &= \frac{1}{\varepsilon}A_{i3}z_i + \frac{1}{\varepsilon}A_{i4}\zeta_i + \varepsilon^{n_i-1}b_i\overline{\mathcal{N}}(\theta_i)\zeta_i + E_{i2}\dot{r}_i \\
\dot{\theta}_i &= \zeta_i^2 \\
\dot{\bar{r}}_1 &= -\alpha M_1^\top(\mathbf{\Pi} + \mathbf{\Pi}_1) \\
\dot{\bar{r}}_2 &= -\alpha M_2^\top(\mathbf{\Pi} + \mathbf{\Pi}_1) - \beta M_L\bar{r}_2 + \alpha M_L\bar{r}_2 - M_L\bar{v}_2 \\
\dot{\bar{v}}_2 &= -\alpha M_L\bar{v}_2 + \alpha^2 M_L\bar{r}_2 - \alpha^2 M_2^\top(\mathbf{\Pi} + \mathbf{\Pi}_1)
\end{aligned}$$

with $\dot{\bar{v}}_1 = 0$ and $\mathbf{\Pi}_1 \triangleq \nabla\tilde{f}(y) - \nabla\tilde{f}(r)$. By Assumption 1, $\mathbf{\Pi}$ and $\mathbf{\Pi}_1$ are $\bar{l}$-Lipschitz with respect to $\bar{r}$ and $y - r$, respectively. By definitions, $\dot{r} = M_1\dot{\bar{r}}_1 + M_2\dot{\bar{r}}_2$. Thus, there exist two constants $\bar{c}_1,\, \bar{c}_2 > 0$ such that $\|\dot{r}\|^2 < \bar{c}_1 W_{\mathrm{o}}(\bar{r},\, \bar{v}_2) + \bar{c}_2\|y - r\|^2$.

Using similar arguments as in the proof of Theorem 1, we take the time derivative of V_i and obtain

$$\begin{aligned}
\dot{V}_i &\le -\frac{2}{\varepsilon}\|z_i\|^2 + \frac{1}{3\varepsilon}\|z_i\|^2 + \frac{3}{\varepsilon}\|P_iA_{i2}\|^2\zeta_i^2 + \frac{1}{3}\|z_i\|^2 \\
&\quad + 3\|P_iE_{i1}\|^2\dot{r}_i^2 + \frac{1}{3\varepsilon}\|z_i\|^2 + \frac{3}{\varepsilon}\|A_{i3}\|^2\zeta_i^2 + \frac{1}{\varepsilon}A_{i4}\zeta_i^2
\end{aligned}$$

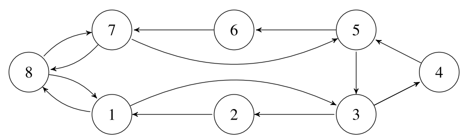

Fig. 1. Interaction graph $\mathcal{G}$ in our examples.

$$\begin{aligned} &+ \varepsilon^{n_i-1} b_i \overline{\mathcal{N}}(\theta_i)\zeta_i^2 + \zeta_i^2 + \|E_{i2}\|^2 \dot{r}_i^2 \\ &= -(\frac{4}{3\varepsilon} - \frac{1}{3})\|z_i\|^2 + (\varepsilon^{n_i-1} b_i \overline{\mathcal{N}}(\theta_i) + \overline{C}_{i\theta_1})\zeta_i^2 + \overline{C}_{i\theta_2}\dot{r}_i^2 \end{aligned}$$

with $\overline{C}_{i\theta_1} = \frac{1}{\varepsilon}(3\|P_i A_{i2}\|^2 + 3\|A_{i3}\|^2 + A_{i4}) + 1$ and $\overline{C}_{i\theta_2} = 3\|P_i E_{i1}\|^2 + \|E_{i2}\|^2$. Differently from the proof of Theorem 1, we avoid ε here in handling the cross terms with $\dot{r}_i$ in order to dominate them by deceasing ε.

Set $z = \mathrm{col}(z_1, \ldots, z_N)$ and $\hat{\overline{C}}_{\theta_2} = \max_{i\in\mathcal{N}}\{\overline{C}_{i\theta_2}\}$. We let $\overline{V} = \sum_{i=1}^N V_i + \sigma W_{\mathrm{o}}$ with $\sigma > 0$ to be specified later. Its time derivative along the trajectory of the error system satisfies

$$\begin{aligned} \dot{\overline{V}} &\le \sum_{i=1}^N [-(\frac{4}{3\varepsilon} - \frac{1}{3})\|z_i\|^2 + (\varepsilon^{n_i-1} b_i \overline{\mathcal{N}}(\theta_i) + \overline{C}_{i\theta_1})\zeta_i^2 + \overline{C}_{i\theta_2}\dot{r}_i^2] \\ &\quad + \sigma\{-\frac{1}{2}W_{\mathrm{o}} - \alpha \bar{r}^\top [M_1\ M_2]^\top \mathbf{\Pi}_1 - \alpha^2 \bar{v}_2^\top M_2^\top \mathbf{\Pi}_1\} \\ &\le -[\frac{4}{3\varepsilon} - \frac{1}{3} - 2\sigma \bar{l}(\alpha^2 + \alpha^7) - \hat{\overline{C}}_{\theta_2}\bar{c}_2]\|z\|^2 - (\frac{\sigma}{4} - \hat{\overline{C}}_{\theta_2}\bar{c}_1)W_{\mathrm{o}} \\ &\quad + \sum_{i=1}^N (\varepsilon^{n_i-1} b_i \overline{\mathcal{N}}(\theta_i) + \overline{C}_{i\theta_1})\zeta_i^2 \end{aligned}$$

Letting $\sigma \ge 8\max\{1, \hat{\overline{C}}_{\theta_2}\bar{c}_1\}$, $\varepsilon^\star = \frac{1}{2\sigma\bar{l}(\alpha^2+\alpha^7) + \hat{\overline{C}}_{\theta_2}\bar{c}_2 + 1}$, and $0 < \varepsilon < \varepsilon^\star$ gives

$$\dot{\overline{V}} \le -\|z\|^2 - W_{\mathrm{o}} + \sum_{i=1}^N (\varepsilon^{n_i-1} b_i \overline{\mathcal{N}}(\theta_i) + \overline{C}_{i\theta_1})\dot{\theta}_i$$

By [2, Lemma 4.4], $\overline{V}(t)$ and $\theta_i(t)$ are bounded over $[0, \infty)$. Then, we can confirm the boundedness of all trajectories of this multi-agent system. Integrating both sides of the above inequality, one can further obtain that $\|z(t)\|^2$ and $W_{\mathrm{o}}(t)$ are both integrable over $[0, \infty)$. Recalling [9, Lemma 8.2], we have $z(t) \to 0$ and $W_{\mathrm{o}} \to 0$ as t goes to ∞. The rest of the proof follows as in Theorem 1. ■

Remark 2: In contrast with most optimal consensus works, we focus on the case where all agents have unknown control directions. Multiple Nussbaum gains are employed in controllers (8) and (10) to overcome the technical difficulties brought by such type of system uncertainties.

Remark 3: Compared with the previous consensus results for multi-agent systems with or without unknown control directions in [7], [14], [16], [18], [24], an optimization requirement is further considered in our formulation. Moreover, by letting $f_i(y) = (y - y_i(0))^2$, these two theorems provide an alternative way to achieve an average consensus goal even when these agents have unknown control directions.

V. Simulation

In this section, we propose two numerical examples to validate the previous theoretical results.

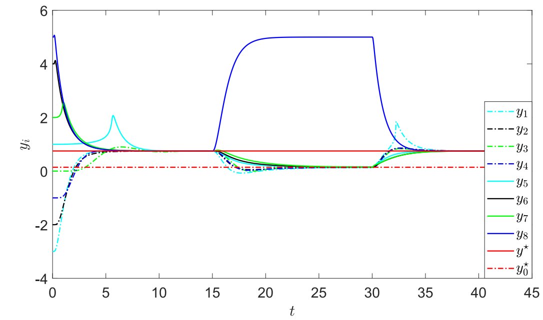

Fig. 2. Profiles of agent outputs in *Example 1*.

Example 1: Consider the average consensus problem for a group of double-integrator agents of the form:

$$\ddot{y}_i = b_i u_i, \quad i = 1, \ldots, 8$$

Assume their interaction topology is depicted in Fig. 1 with unit weights. Assumption 2 can be verified.

According to Remark 3, we can let $f_i(y) = (y - y_i(0))^2$ for $i = 1, \ldots, 8$ and use the controller (8) with $n_i = 2$ to complete the design. For simulation, we set $b_1 = \cdots = b_4 = -1$, $b_5 = \cdots = b_8 = 1$, and $y(0) = [-3\ \ -2\ 0\ \ -1\ 1\ 4\ 2\ 5]^\top$. Distributed controller (8) with $\varepsilon = 1$, $k_{i1} = 1$ for $i = 1, \ldots, 8$, and $\overline{\mathcal{N}}(\theta) = \theta^2 \sin\theta$ is then applied to solve this problem. To make it more interesting, we cut all links associated with node 8 at $t = 15$s and then add them back at $t = 30$s. The simulation result is depicted in Fig. 2. At first, the outputs of agents are observed to reach an average consensus on $y^\star = \frac{\sum_{i=1}^8 y_i(0)}{8} = 0.75$. Then, $y_8(t)$ converges to its local optimizer $y_8(0) = 5$ while the other agents reach a consensus on $y_0^\star = \frac{\sum_{i=1}^7 y_i(0)}{7} = 0.143$. After the links are added back, the average consensus for all agents is quickly recovered at $y^\star$. This verifies the robustness of our algorithms enabling plug-and-play operations.

Example 2: Consider the optimal consensus problem for a heterogeneous multi-agent system described by

$$y_i^{(n_i)} = b_i u_i, \quad i = 1, \ldots, 8$$

with the same topology as that in *Example 1*. Here, $n_1 = n_5 = 1$, $n_2 = n_6 = 2$, $n_3 = n_7 = 3$, and $n_4 = n_8 = 4$.

The local cost functions are taken as $f_1(y) = f_5(y) = (y-8)^2$, $f_2(y) = f_6(y) = \frac{y^2}{20\sqrt{y^2+1}} + y^2$, $f_3(y) = f_7(y) = \frac{y^2}{80\ln(y^2+2)} + (y-5)^2$, $f_4(y) = f_8(y) = \ln(e^{-0.05y} + e^{0.05y}) + y^2$. Assumption 1 holds for $\underline{l} = 1$, $\bar{l} = 3$ as shown in [26]. Moreover, the global optimal point can be obtained numerically as $y^\star = 3.24$. Since these agents are of heterogeneous orders and unknown high-frequency gains, the rules developed in [23], [33] fail to tackle this problem. Nevertheless, according to Theorems 1 and 2, we can utilize controller (8) or (10) to solve it.

For simulation, we let $b_1 = \cdots = b_8 = -1$. Choose $k_{21} = k_{61} = 1$, $k_{31} = k_{71} = 1$, $k_{32} = k_{72} = 2$, $k_{41} = k_{81} = 1$, $k_{42} = k_{82} = 3$, $k_{43} = k_{83} = 3$, $\varepsilon = 0.5$, and $\overline{\mathcal{N}}(\theta) = e^{\theta^2}\sin\theta$ for controller (10). To verify the robustness of our algorithm, we add an actuated disturbance $10\sin(t)$ to all agents during $15\mathrm{s} \le t \le 30\mathrm{s}$. The simulation result is depicted in Figs. 3 and 4. One can observe that all agents quickly reach an optimal consensus on $y^\star = 3.24$ at first while the profiles of agents'

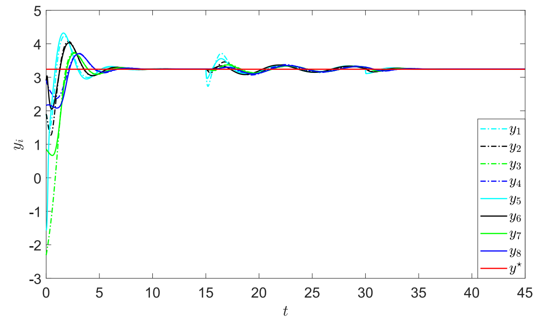

Fig. 3. Profiles of agent outputs in *Example 2*.

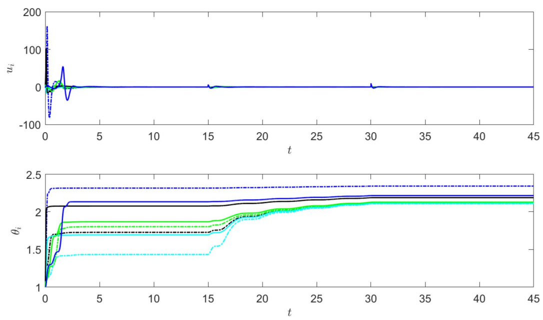

Fig. 4. Profiles of control efforts and adaptive gains in *Example 2*.

control efforts are maintained bounded. Then, the expected exact optimal consensus is broken due to actuated disturbances but the error $|y_i - y^\star|$ is still bounded. These observations verify the efficacy and robustness of our adaptive optimal consensus algorithms in handling both heterogeneous agent dynamics and unknown control directions.

VI. Conclusion

An optimal consensus problem has been discussed for a high-order multi-agent system without a prior knowledge of the control directions. By an embedded control design, we have proposed two Nussbaum-type distributed controllers to solve it under different information circumstances. Further works will include improvement of transient performances and extensions to more general agent dynamics.

References

[1] W. Chen, X. Li, W. Ren, and C. Wen, "Adaptive consensus of multi-agent systems with unknown identical control directions based on a novel Nussbaum-type function," *IEEE Trans. Autom. Control*, vol. 59, no. 7, pp. 1887–1892, Jul. 2014.

[2] Z. Chen, "Nussbaum functions in adaptive control with time-varying unknown control coefficients," *Automatica*, vol. 102, pp. 72–79, Apr. 2019.

[3] Z. Ding, "Adaptive consensus output regulation of a class of nonlinear systems with unknown high-frequency gain," *Automatica*, vol. 51, pp. 348–355, Jan. 2015.

[4] J. Du, C. Guo, S. Yu, and Y. Zhao, "Adaptive autopilot design of time-varying uncertain ships with completely unknown control coefficient," *IEEE J. Ocean. Eng*, vol. 32, no. 2, pp. 346–352, Apr. 2007.

[5] C. Godsil and G. Royle, *Algebraic Graph Theory*. New York, NY, USA: Springer, 2001.

[6] M. Guo, D. Xu, and L. Liu, "Cooperative output regulation of heterogeneous nonlinear multi-agent systems with unknown control directions," *IEEE Trans. Autom. Control*, vol. 62, no. 6, pp. 3039–3045, Jun. 2017.

[7] J. Huang, Y. Song, W. Wang, C. Wen, and G. Li, "Fully distributed adaptive consensus control of a class of high-order nonlinear systems with a directed topology and unknown control directions," *IEEE Trans. Cybern.*, vol. 48, no. 8, pp. 2349–2356, Aug. 2018.

[8] D. Jakovetić, J. M. Moura, and J. Xavier, "Linear convergence rate of a class of distributed augmented Lagrangian algorithms," *IEEE Trans. Autom. Control*, vol. 60, no. 4, pp. 922–936, Apr. 2015.

[9] H. K. Khalil, *Nonlinear Systems (3rd ed.)*. Upper Saddle River, NJ, USA: Prentice-Hall, 2002.

[10] S. S. Kia, J. Cortés, and S. Martínez, "Distributed convex optimization via continuous-time coordination algorithms with discrete-time communication," *Automatica*, vol. 55, pp. 254–264, May 2015.

[11] Y. Liu and G. Tao, "Multivariable MRAC using nussbaum gains for aircraft with abrupt damages," in *Proc. 47th IEEE Conf. Decis. Control*, Cancún, Mexico, 2008, pp. 2600–2605.

[12] A. Nedić and J. Liu, "Distributed optimization for control," *Annu. Rev. Control Robot. Auton. Syst.*, vol. 1, pp. 77–103, May 2018.

[13] R. D. Nussbaum, "Some remarks on a conjecture in parameter adaptive control," *Syst. Control Lett.*, vol. 3, no. 5, pp. 243–246, 1983.

[14] J. Peng and X. Ye, "Cooperative control of multiple heterogeneous agents with unknown high-frequency-gain signs," *Syst. Control Lett.*, vol. 68, pp. 51–56, Jun. 2014.

[15] Z. Qiu, L. Xie, and Y. Hong, "Distributed optimal consensus of multiple double integrators under bounded velocity and acceleration," *Control Theory Technol.*, vol. 17, no. 1, pp. 85–98, 2019.

[16] W. Ren and R. Beard, *Distributed Consensus in Multi-vehicle Cooperative Control: Theory and Applications*. London, U.K.: Springer, 2008.

[17] H. Rezaee and F. Abdollahi, "Average consensus over high-order multiagent systems," *IEEE Trans. Autom. Control*, vol. 60, no. 11, pp. 3047–3052, Nov. 2015.

[18] M. H. Rezaei, M. Kabiri, and M. B. Menhaj, "Adaptive consensus for high-order unknown nonlinear multi-agent systems with unknown control directions and switching topologies," *Inform. Sci.*, vol. 459, pp. 224–237, Aug. 2018.

[19] A. Ruszczynski, *Nonlinear Optimization*. Princeton, NJ, USA: Princeton Univ. Press, 2011.

[20] G. Shi, K. H. Johansson, and Y. Hong, "Reaching an optimal consensus: Dynamical systems that compute intersections of convex sets," *IEEE Trans. Autom. Control*, vol. 58, no. 3, pp. 610–622, Mar. 2013.

[21] W. Song, Y. Tang, Y. Hong, and X. Hu, "Relative attitude formation control of multi-agent systems," *Int. J. Robust Nonlinear Control*, vol. 27, no. 18, pp. 4457–4477, 2017.

[22] E. D. Sontag, "A remark on the converging-input converging-state property," *IEEE Trans. Autom. Control*, vol. 48, no. 2, pp. 313–314, Feb. 2003.

[23] Y. Tang, Z. Deng, and Y. Hong, "Optimal output consensus of high-order multiagent systems with embedded technique," *IEEE Trans. Cybern.*, vol. 49, no. 5, pp. 1768–1779, May 2019.

[24] Y. Tang, "Output consensus of nonlinear multi-agent systems with unknown control directions," *Kybernetika*, vol. 51, no. 2, pp. 335–346, 2015.

[25] Y. Tang, "Distributed optimization for a class of high-order nonlinear multiagent systems with unknown dynamics," *Int. J. Robust Nonlinear Control*, vol. 28, no. 17, pp. 5545–5556, 2018.

[26] Y. Tang and X. Wang, "Optimal output consensus for nonlinear multi-agent systems with both static and dynamic uncertainties," *IEEE Trans. Autom. Control*, early access, May 25, 2020, [Online]. Available: http://ieeexplore.ieee.org/abstract/document/9099468

[27] X. Wang, Y. Hong, and H. Ji, "Distributed optimization for a class of nonlinear multiagent systems with disturbance rejection," *IEEE Trans. Cybern.*, vol. 46, no. 7, pp. 1655–1666, Jul. 2016.

[28] Y. Xie and Z. Lin, "Global optimal consensus for multi-agent systems with bounded controls," *Syst. Control Lett.*, vol. 102, pp. 104–111, Apr. 2017.

[29] T. Yang *et al.*, "A survey of distributed optimization," *Annu. Rev. Control*, vol. 47, pp. 278–305, May 2019.

[30] X. Ye and J. Jiang, "Adaptive nonlinear design without a priori knowledge of control directions," *IEEE Trans. Autom. Control*, vol. 43, no. 11, pp. 1617–1621, Nov. 1998.

[31] C. Zhang and R. Ordóñez, *Extremum-Seeking Control and Applications: A Numerical Optimization-based Approach*. London, U.K.: Springer, 2011.

[32] X. Zhang, A. Papachristodoulou, and N. Li, "Distributed control for reaching optimal steady state in network systems: An optimization approach," *IEEE Trans. Autom. Control*, vol. 63, no. 3, pp. 864–871, Mar. 2018.

[33] Y. Zhang, Z. Deng, and Y. Hong, "Distributed optimal coordination for multiple heterogeneous Euler–Lagrangian systems," *Automatica*, vol. 79, pp. 207–213, May 2017.