

Optimal Output Consensus of High-Order Multiagent Systems With Embedded Technique

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Abstract—In this paper, we study an optimal output consensus problem for a multiagent network with agents in the form of multi-input multioutput minimum-phase dynamics. Optimal output consensus can be taken as an extended version of the existing output consensus problem for higher-order agents with an optimization requirement, where the output variables of agents are driven to achieve a consensus on the optimal solution of a global cost function. To solve this problem, we first construct an optimal signal generator, and then propose an embedded control scheme by embedding the generator in the feedback loop. We give two kinds of algorithms based on different available information along with both state feedback and output feedback, and prove that these algorithms with the embedded technique can guarantee the solvability of the problem for high-order multiagent systems under standard assumptions.

Index Terms—Embedded control, high-order system, optimal output consensus, optimal signal generator.

I. INTRODUCTION

COORDINATION problems of multiagent systems have drawn much research interests due to the fast development of large-scale systems/networks, and multiple high-order agents have been widely discussed to deal with some practical coordination problems, including consensus and formation in recent years [1]–[4]. Particularly, because of the cost or difficulty in the measurement of all the agent states, output consensus for high-order (minimum-phase) agents has been widely studied [5]–[9].

In some practical applications, it might not be enough to achieve only a consensus in multiagent systems and optimality issues of the consensus point have to be taken into consideration. Recently, distributed optimization has attracted more and more attention with their broad potential applications in multiagent systems, smart grid, and sensor networks [10]–[13].

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Among them, the case with a sum of convex objective functions has been intensively studied these years. In the problem setup, each agent is assigned with a local cost function and the control objective is to propose distributed control that guarantees a consensus on the optimal solution of the sum of all local cost functions. Many results were obtained based on gradients or subgradients of the local cost functions combined with consensus rules. In addition to many discrete-time algorithms [10], [14]–[16], continuous-time gradient-based optimal consensus algorithms were proposed in different situations, considering that there are many powerful continuous-time methods. For example, Shi *et al.* [17] reduced the distributed optimization problem as an intersection computing problem of convex sets, where each convex set is determined by the local cost function. An alternative continuous-time algorithm was provided with discrete-time communication in [18] to solve this problem, and moreover, a distributed optimization algorithm with disturbance rejection was studied by virtue of internal model principle in [19]. In addition, various gradient-based algorithms have also been proposed to handle the constrained distributed optimization problem (refer to [20]–[24]). Note that most of the literature only consider single-integrator agents.

Furthermore, many optimal consensus or distributed optimization tasks may be implemented or depend on physical dynamics in practice, thus the distributed design involved with continuous-time agents dynamics has to be taken into account. In fact, continuous-time physical systems or hybrid cyber-physical systems are employed in practical coordination tasks, for example, in cooperative search of radio sources [25] and in the distributed optimal power flow problem [12], [26]. Since practical systems are hardly described by single integrators, we may have to take high-order dynamics into consideration. For example, for double integrators, Zhang and Hong [27] proposed a distributed optimization algorithm with an integral control idea and a similar design was employed for Lagrangian agents [28]. It was later extended to high-order integrators [29] assumed the existence of a kind of Lyapunov functions for high-order agents. Generally, this problem is still far from being solved.

The objective of this paper is to study the optimal output consensus problem for high-order minimum-phase multiagent systems. To be specific, we give a framework for the optimal output consensus design over a group of multi-input multioutput (MIMO) minimum-phase agents. In light of the optimization requirement for output consensus, the gradient-based closed-loop systems are basically nonlinear since we

consider general convex objective functions, although the agents are in the form of linear dynamics. The high-order features of these agents bring many new problems in its analysis and design, which make this problem much more challenging.

To overcome the difficulties brought by both high-order dynamics and nonlinearities from gradients, we propose an embedded control scheme, consisting of an optimal signal generator and a reference-tracking controller. This embedded technique makes the control design carried out in a “separative” way, so as to simplify the whole design by almost independently tackling the optimal consensus problem for single integrators and tracking control of high-order agents. In fact, the main contributions of this paper include the following.

- 1) A general framework of optimal output consensus was formulated and solved for a group of MIMO minimum-phase agents with vector relative degrees. This problem can be taken as an extension by combining the two hot topics: a) output consensus of high-order agents [6]–[9] and b) distributed optimization for single integrators [10], [15]–[18]. Since output consensus must be achieved as the solution of a convex optimization problem of the whole multiagent system, it is certainly more challenging than the existing output consensus problem for high-order agents. Our optimal output consensus results, in either asymptotic or exponential stability, are consistent with the existing results.
- 2) A novel embedded control method is proposed for this optimal output consensus problem to handle the high-order dynamics in multiagent systems. By constructing an optimal signal generator and embedding it in the feedback loop, we can solve this problem by almost independently achieving an optimal consensus over these generators and reference-tracking control of high-order agents. Compared with existing optimization results for high-order integrators with some Lyapunov function assumptions in [29], this embedded technique provides a quite general and constructive way to solve this problem via separating the optimization and control design.
- 3) Two distributed gradient-based algorithms are provided by the given embedded approach in two different gradient information cases. In addition to state feedback algorithms for this problem, output feedback extensions with local observer design in each case are also proposed since the state variable of high-order agents may not be available by measurements. Under some standard assumptions, the optimal output consensus can be achieved asymptotically and even exponentially, while many existing continuous-time publications are on asymptotic stability [17], [23], [29].

The organization of this paper is as follows. Preliminaries about convex analysis and graph theory are introduced and the optimal output consensus problem is formulated in Section II. Then an embedded optimization control framework is proposed for high-order minimum-phase agents with vector relative degrees in Section III. Main results are presented and proved in Section IV along with the given gradient-based

control in both state feedback and output feedback cases. Following that, two examples are given to illustrate the effectiveness of the proposed algorithm in Section V. Finally, concluding remarks are given in Section VI.

Notations: Let \mathbb{R}^n be the n -dimensional Euclidean space. For a vector x , $\|x\|$ denotes its Euclidean norm. $\mathbf{1}_N$ (and $\mathbf{0}_N$) denotes an N -dimensional all-one (and all-zero) column vector. $\text{col}(a_1, \dots, a_n) = [a_1^\top, \dots, a_n^\top]^\top$ for column vectors a_i ($i = 1, \dots, n$). $r_N = (1/\sqrt{N})\mathbf{1}_N$, and $R_N \in \mathbb{R}^{N \times (N-1)}$ satisfying $R_N^\top r_N = \mathbf{0}_N$, $R_N^\top R_N = I_{N-1}$ and $R_N R_N^\top = I_N - r_N r_N^\top$.

II. PRELIMINARIES

In this section, some basic concepts are introduced for convex analysis [30], [31] and graph theory [32], and then the optimal output consensus problem is formulated.

A. Convex Analysis

A function $f(\cdot) : \mathbb{R}^m \rightarrow \mathbb{R}$ is said to be convex if, for any $0 \leq a \leq 1$

$$f(a\zeta_1 + (1-a)\zeta_2) \leq af(\zeta_1) + (1-a)f(\zeta_2), \quad \forall \zeta_1, \zeta_2 \in \mathbb{R}^m.$$

A differentiable function f is convex over \mathbb{R}^m if

$$f(\zeta_1) - f(\zeta_2) \geq \nabla f(\zeta_2)^\top (\zeta_1 - \zeta_2), \quad \forall \zeta_1, \zeta_2 \in \mathbb{R}^m \quad (1)$$

and f is strictly convex over \mathbb{R}^m if the above inequality is strict whenever $\zeta_1 \neq \zeta_2$, and f is ω -strongly convex ($\omega > 0$) over \mathbb{R}^m if $\forall \zeta_1, \zeta_2 \in \mathbb{R}^m$

$$(\nabla f(\zeta_1) - \nabla f(\zeta_2))^\top (\zeta_1 - \zeta_2) \geq \omega \|\zeta_1 - \zeta_2\|^2. \quad (2)$$

A vector-valued function $\mathbf{f} : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is Lipschitz with constant $\vartheta > 0$, or simply ϑ -Lipschitz, if

$$\|\mathbf{f}(\zeta_1) - \mathbf{f}(\zeta_2)\| \leq \vartheta \|\zeta_1 - \zeta_2\|, \quad \forall \zeta_1, \zeta_2 \in \mathbb{R}^m.$$

B. Graph Theory

A weighted undirected graph is described by $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$ with the node set $\mathcal{N} = \{1, \dots, N\}$ and the edge set \mathcal{E} (without self-loops). $(i, j) \in \mathcal{E}$ denotes an edge between nodes i and j . The weighted adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ is defined by $a_{ii} = 0$ and $a_{ij} = a_{ji} > 0$ ($a_{ij} > 0$ if and only if there is an edge between node i and node j). The neighbor set of node i is defined as $\mathcal{N}_i = \{j : (j, i) \in \mathcal{E}\}$ for $i = 1, \dots, n$. A path in graph \mathcal{G} is an alternating sequence $i_1 e_1 i_2 e_2 \dots e_{k-1} i_k$ of nodes i_l and edges $e_m = (i_m, i_{m+1}) \in \mathcal{E}$ for $l = 1, 2, \dots, k$. If there is a path between any two vertices, then the graph is said to be connected. The Laplacian $L = [l_{ij}] \in \mathbb{R}^{N \times N}$ of graph \mathcal{G} is defined as $l_{ii} = \sum_{j \neq i} a_{ij}$ and $l_{ij} = -a_{ij}$ ($j \neq i$), which is thus symmetric. Denote the eigenvalues of Laplacian matrix L associated with a undirected graph \mathcal{G} as $\lambda_1 \leq \dots \leq \lambda_N$. The following lemma is well-known [32].

Lemma 1: $\lambda_1 = 0$ is an eigenvalue of L with $\mathbf{1}_N$ as its corresponding eigenvector, and $\lambda_2 > 0$ if and only if the graph \mathcal{G} is connected.

C. Problem Formulation

In this paper, we consider a group of N continuous-time linear agents as follows:

$$\begin{cases} \dot{\tilde{x}}_i = \tilde{A}\tilde{x}_i + \tilde{B}\tilde{u}_i \\ y_i = \tilde{C}\tilde{x}_i, \quad i = 1, 2, \dots, N \end{cases} \quad (3)$$

where $\tilde{x}_i \in \mathbb{R}^k$, $y_i \in \mathbb{R}^m$, and $\tilde{u}_i \in \mathbb{R}^m$ are the state variable, the output variable and the control input, respectively.

Vector relative degree is a well-known concept and widely used in the study of MIMO systems, which is introduced as follows.

Definition 1: System (3) is said to have a vector relative degree (r_1, \dots, r_m) if:

- 1) $\tilde{C}_i \tilde{A}^k \tilde{B}_j = 0$ for all $1 \leq i, j \leq m$ and $0 \leq k < r_i - 1$;
- 2) the $m \times m$ matrix $\tilde{R} \triangleq [t_{ij}]$ is nonsingular with $t_{ij} = \tilde{C}_i \tilde{A}^{r_i-1} \tilde{B}_j$, where \tilde{C}_i is the i th row of matrix \tilde{C} and \tilde{B}_j is the j th column of \tilde{B} .

To clarify our optimal output consensus problem, we focus on a class of systems as follows.

Assumption 1: System (3) is minimum-phase and has a vector relative degree (r_1, \dots, r_m) .

Remark 1: In fact, this assumption has been frequently used in the study of linear or nonlinear MIMO systems [33], [34] including integrators as special cases. Clearly, a single-input single-output linear system certainly has a relative degree.

Associated with these agents, the communication topology can be described by a weighted undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where an edge $(i, j) \in \mathcal{E}$ means that agent i and j can exchange information each other. Furthermore, each agent is endowed with a differentiable local cost function $f_i: \mathbb{R}^m \rightarrow \mathbb{R}$. The global cost function is defined as the sum of local costs, i.e., $f(y) = \sum_{i=1}^N f_i(y)$. In distributed design, agent i has its own local cost function $f_i(\cdot)$, which is only known to itself and cannot be shared globally in the multiagent network.

The optimal output consensus problem associated with those agents (3) is then formulated as follows. Given a multiagent system (3), the communication graph \mathcal{G} and local cost function $f_i(\cdot)$, find a distributed control \tilde{u}_i for agent i by only its own local data and exchanged information with its neighbors such that all the trajectories of agents are bounded and their outputs satisfy

$$\lim_{t \rightarrow +\infty} \|y_i(t) - y^*\| = 0 \quad (4)$$

where y^* is the optimal solution of

$$\min_{y \in \mathbb{R}^m} f(y). \quad (5)$$

Remark 2: Different from the widely studied output consensus problem, optimal output consensus we formulated for high-order agents can be regarded as its extension by requiring further the consensus point as the optimal solution of a convex cost function. In fact, as the requirement (4) coincides distributed optimization problem when agents are single integrators, it apparently implies an output consensus of the multiagent systems [7], [9]. Many practical problems can be written in this form, including motion planning and formation control in robotics [35], [36]. As optimal output consensus focuses on static optimization and the steady-state

performance of high-order dynamic agents, we do not give any limitations or concerns about the energy or input when we achieve the output consensus. In other words, energy/input optimization is usually involved with optimal control, but here our optimal output consensus cares about the output consensus point as the solution of a convex optimization.

The following assumptions are often made to solve output consensus and/or distributed optimization problems [7], [18], [19], [37].

Assumption 2: The graph \mathcal{G} is undirected and connected.

Assumption 3: For $i = 1, \dots, N$, the function $f_i(\cdot)$ is strictly convex.

Assumption 2 is about the connectivity of graph \mathcal{G} , which guarantees that any agent's information can reach any other agents. Assumption 3 implies the uniqueness of the optimal solution to (5) [30]. As usual, we assume there exists a finite optimal solution to (5) (refer to [18] and [27]). To get better convergence performances, we make another assumption as follows.

Assumption 4: For $i = 1, \dots, N$, the function $f_i(\cdot)$ is ω -strongly convex and its gradient is ϑ -Lipschitz for some positive constants ω and ϑ .

Assumption 4 is stronger than Assumption 3. In fact, strong convexity implies the strict convexity and the Lipschitz condition here facilitates the study of exponential convergence. Both the two assumptions have been widely used in (distributed) optimization problem [18], [19], [23], [37].

Note that the gradient-based optimization control is basically nonlinear and the general high-order system structure needs an effective design policy to optimize the output variables, which make the optimal output consensus problem much more challenging than most existing optimal consensus designs for the single-integrator and double-integrator agents. To tackle these problems, we propose an embedded control framework in the next section.

III. EMBEDDED CONTROL SCHEME

In this section, we introduce an embedded control framework to solve the optimal output consensus problem of high-order MIMO agents in a quite unified way.

Embedded ideas or techniques are becoming ubiquitous in control applications [38]. They provide flexible and reconfigurable structures to design controllers for complicated systems by integration and implementation of some standard but simple control designs. In the following, we first brief the whole idea of our embedded control scheme depicted in Fig. 1 and then present the first two steps of this approach.

This scheme consists of three components: 1) precompensator to deal with vector relative degree for (3); 2) optimal signal generator to solve (5); and 3) reference-tracking controller to force the agent to track the generator. With a precompensator, the original system is transformed to a normal form (10a) with homogeneous relative degrees. This simplifies the design and makes us only focus on high-order integrators. To avoid the difficulties resulting from the high-order structure, we introduce an optimal signal generator by considering the same optimization problem for "virtual" single integrators,

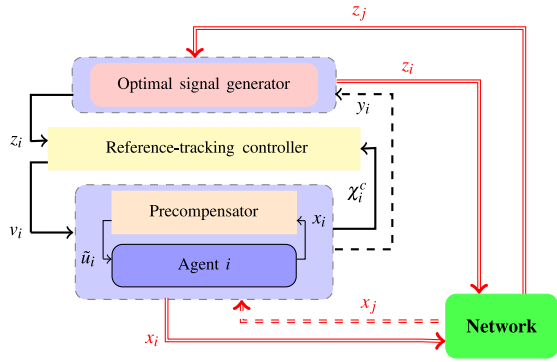


Fig. 1. Schematic of the embedded control framework in optimal output consensus.

in order to asymptotically reproduce the optimal solution y^* by a signal z_i . Then by taking z_i as an output reference signal for the high-order agents and embedding this generator in the feedback loop via a tracking controller to be designed in the next section, we solve the optimal output consensus problem of system (3).

Note that the key part for the optimization is the optimal signal generator, embedded in the feedback loop so that the physical system can achieve the optimization after its output follows the optimal signal given by this generator. In this way, we first solve the optimal (output) consensus problem for simple integrators, and then seek to synthesize the controllers for physical complex agents by embedding the optimal signal generator in a well-designed reference-tracking controller for the system (10a) to follow (16). Detailed designs of the precompensator and optimal signal generator are given in the following two sections.

A. Precompensator

Many high-order agents have vector relative degrees with $r_i \neq r_j$ when $i \neq j$. To make the design simpler, we make a precondition on agents' dynamics to achieve input-output decoupling with the same relative degree in each channel. To be specific, we transform (3) by precompensation techniques [33] to a normal form through decoupling the MIMO agent dynamics and homogenizing the relative degrees.

According to [34] and [39], we can determine a group of basis under Assumption 1 and the associated coordinate transformation as follows:

$$\begin{cases} \xi_{ij}^{ib} = \tilde{C}_l \tilde{A}^{i-1} \tilde{x}_i \text{ with } j = 1, \dots, r_l \text{ and } l = 1, \dots, m \\ \xi_{i\kappa}^{ia} = \psi_i \tilde{x} \text{ with } \psi_i \tilde{B} = \mathbf{0}, l = 1, \dots, \kappa - (r_1 + \dots + r_m). \end{cases}$$

Denote $\chi_i^b = \text{col}(\xi_{11}^{ib}, \dots, \xi_{1r_1}^{ib}, \dots, \xi_{m1}^{ib}, \dots, \xi_{mr_m}^{ib})$ and $\chi_i^a = \text{col}(\xi_1^{ia}, \dots, \xi_{\kappa - \sum_{j=1}^m r_j}^{ia})$. Then the system (3) can be written in the following form:

$$\begin{cases} \dot{\chi}_i^a = \Pi \chi_i^a + \Psi \chi_i^b \\ \dot{\xi}_{ij}^{ib} = \xi_{i(j+1)}^{ib}, j = 1, \dots, r_l - 1 \\ \dot{\xi}_{lr_l}^{ib} = \Upsilon_l \chi_i^a + S_l \chi_i^b + \tilde{C}_l \tilde{A}^{r_l-1} \tilde{B} \tilde{u}_i \\ y_i = \text{col}(\xi_{11}^{ib}, \dots, \xi_{m1}^{ib}), l = 1, \dots, m, i = 1, \dots, N \end{cases} \quad (6)$$

where $\Upsilon_l \in \mathbb{R}^{\kappa - \sum_{j=1}^m r_j}$, $S_l \in \mathbb{R}^{\sum_{j=1}^m r_j}$, $\Psi \in \mathbb{R}^{(\kappa - \sum_{j=1}^m r_j) \times \sum_{j=1}^m r_j}$, and $\Pi \in \mathbb{R}^{(\kappa - \sum_{j=1}^m r_j) \times (\kappa - \sum_{j=1}^m r_j)}$. Note

that Π is Hurwitz (that is, the real parts of its eigenvalues are negative) due to the minimum-phase of system (3).

Take $u_i = \text{col}(u_{i1}, \dots, u_{im})$, $\Upsilon = \text{col}(\Upsilon_1, \dots, \Upsilon_m)$, $S = \text{col}(S_1, \dots, S_m)$, and

$$\tilde{u}_i = \tilde{R}^{-1} \left(-\Upsilon \chi_i^a - S \chi_i^b + u_i \right). \quad (7)$$

Therefore,

$$\begin{cases} \dot{\chi}_i^a = \Pi \chi_i^a + \Psi \chi_i^b \\ \dot{\xi}_{ij}^{ib} = \xi_{i(j+1)}^{ib}, j = 1, \dots, r_l - 1 \\ \dot{\xi}_{lr_l}^{ib} = u_{il} \\ y_i = \text{col}(\xi_{11}^{ib}, \dots, \xi_{m1}^{ib}), l = 1, \dots, m, i = 1, \dots, N. \end{cases} \quad (8)$$

Without loss of generality, we assume $\max\{r_1, \dots, r_m\} = n$. Then u_i is taken as

$$\begin{cases} u_{il}^{(n-r_l)} = v_{il}, \text{ when } n > r_l \\ u_{il} = v_{il}, \text{ when } n = r_l, l = 1, \dots, m \end{cases} \quad (9)$$

where $u_{il}^{(n-r_l)}$ is the $(n - r_l)$ th derivative of u_{il} . This implies

$$\begin{cases} \dot{\chi}_i^a = \Pi \chi_i^a + \Psi \chi_i^b \end{cases} \quad (10a)$$

$$\begin{cases} \dot{\chi}_i^c = (A \otimes I_m) \chi_i^c + (b \otimes I_m) v_i \end{cases} \quad (10b)$$

$$\begin{cases} y_i = (c \otimes I_m) \chi_i^c, i = 1, \dots, N \end{cases} \quad (10c)$$

where $\chi_i^c = \text{col}(y_i, \dots, y_i^{(n-1)})$, $y_i^{(j)} = \text{col}(y_{i1}^{(j)}, \dots, y_{im}^{(j)})$

$$y_{il}^{(j)} = \begin{cases} \xi_{i(j+1)}^{ib}, j \leq r_l - 1 \\ u_{il}, j = r_l \\ u_{il}^{(j-r_l)}, j > r_l \end{cases} \quad l = 1, \dots, m, j = 1, \dots, n - 1$$

and

$$A = \begin{bmatrix} \mathbf{0}_{n-1} & I_{n-1} \\ 0 & \mathbf{0}_{n-1}^\top \end{bmatrix}, b = \begin{bmatrix} \mathbf{0}_{n-1}^\top & 1 \end{bmatrix}^\top, c = \begin{bmatrix} 1 & \mathbf{0}_{n-1}^\top \end{bmatrix}.$$

Consequently, we can rewrite system (3) in a normal form (10a), where the inputs and outputs are decoupled and with homogeneous relative degrees (n, \dots, n) . In this way, we only have to consider agents in an integrator form composed of (10b)–(10c) and get back to linear minimum-phase agents of the form (3) by a combination of (7) and well-designed v_i .

B. Optimal Signal Generator

Clearly, in the optimal output consensus problem, we have two main tasks: 1) output variables of all the agents will go to the common optimal solution and 2) all the state variables will be bounded. Here, we give an embedded technique to “divide” the problem in order to deal with the nontrivial difficulties in the whole design. In other words, we first construct an optimal signal generator to get the optimal consensus for virtual single integrators by leveraging the existing distributed optimization ideas and then use the leader-following ideas to achieve the output consensus for high-order agents, where the design for the virtual single integrators can be viewed as “embedded” technique. Here, we define an optimal signal generator as follows.

Definition 2: A system $\dot{\xi} = g(\xi), z = h(\xi)$ is said to be an optimal signal generator associated with (f_D, S, b) if its

trajectory is well-defined and satisfies $z \rightarrow y^*$ as $t \rightarrow +\infty$ where y^* solves

$$\begin{aligned} \min_{y \in \mathbb{R}^n} f_D(y) \\ \text{s.t. } S^\top y = b \end{aligned} \quad (11)$$

where $S \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

Obviously, an optimal signal generator is a system that can asymptotically reproduce the optimal solution of (11) (for single integrators), and thus the design of optimal signal generator is independent of available information and high-order plants. Although there may be various candidates, some of them may not lead to an augmented system for which the reference-tracking problem can be solved. Hence, considering system composition or embedded design, we also need to concern the robustness of these generators.

Before constructing a suitable optimal signal generator for (5), we present a useful lemma.

Lemma 2: Consider a system of the form

$$\dot{x} = -\phi(x) - Sz, \quad \dot{z} = S^\top x + Tz \quad (12)$$

where $x \in \mathbb{R}^n$, $z \in \mathbb{R}^l$, $\phi: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is smooth and satisfies $\phi(\mathbf{0}_n) = \mathbf{0}_n$. Assume $T + T^\top \leq 0$, (S, T) is observable, and $x^\top \phi(x) > 0$ if $x \neq \mathbf{0}_n$. Then the origin of the system is asymptotically stable. Furthermore, if $\phi(\cdot)$ is ϑ -Lipschitz and satisfies $x^\top \phi(x) \geq \omega x^\top x$ for some positive constants ϑ and ω , then the origin of the system is globally exponentially stable.

Proof: Take a Lyapunov function $V = x^\top x + z^\top z$. It follows that:

$$\dot{V} \leq -2x^\top \phi(x) + z^\top (T + T^\top) z \leq 0. \quad (13)$$

According to the LaSalle's invariance principle [40], every trajectory of this system approaches the largest invariant set contained in $E = \{(x, z) \mid x^\top \phi(x) = 0\}$. Since no trajectory can stay identically in E , other than the trivial solution $(\mathbf{0}_n, \mathbf{0}_l)$ by the strict positiveness of $\phi(\cdot)$ and observability of the pair (S, T) . Hence, the whole system is asymptotically stable at $(\mathbf{0}_n, \mathbf{0}_l)$.

When $\phi(\cdot)$ is ϑ -Lipschitz, from its smoothness, its Jacobian $[(\partial\phi)/(\partial x)]$ is globally bounded. Obviously, $\phi(x) = [\int_0^1 [(\partial\phi)/(\partial x)](\theta x) d\theta] x$ is well-defined. Therefore, there exists a time-dependent matrix $D(t) = \int_0^1 [(\partial\phi)/(\partial x)](\theta x) d\theta$ along the trajectory of system (12), which is clearly uniformly bounded with respect to t . We then represent the system as follows:

$$\dot{x} = -D(t)x - Sz, \quad \dot{z} = S^\top x + Tz. \quad (14)$$

Note that $\bar{A}_o(t) = A_o - C^\top D(t)C$ with

$$\bar{A}_o(t) = \begin{bmatrix} -D(t) & -S \\ S^\top & T \end{bmatrix}, \quad A_o = \begin{bmatrix} \mathbf{0} & -S \\ S^\top & T \end{bmatrix}, \quad C = [I_n \ \mathbf{0}].$$

Recalling the uniform boundedness of $D(t)$ and by [41, Lemma 4.8.1] or following its proofs, we obtain that the pair $(C, \bar{A}_o(t))$ is uniformly completely observable from the observability of (C, A_o) . Hence, there exist positive constants δ and k such that

$$W(t, t + \delta) = \int_t^{t+\delta} \Phi^\top(\tau, t) C^\top C \Phi(\tau, t) d\tau \geq kI_n, \quad \forall t \geq 0$$

where $\Phi(\cdot, \cdot)$ is the state transition matrix of (14).

The derivative of V then satisfies

$$\dot{V} \leq -x^\top \phi(x) \leq -\omega x^\top x. \quad (15)$$

Integrating it from t to $t + \delta$, we have that, for all $t \geq 0$

$$\int_t^{t+\delta} \dot{V}(\tau) d\tau \leq -\omega k (\|x(t)\|^2 + \|z(t)\|^2) = -2\omega k V(t).$$

Based on [40, Th. 8.5], one can conclude the global exponential stability of this system at the origin. ■

Remark 3: An interesting special case is when $T = 0$ and S has a full-column rank. Apparently, (S, T) is observable. If we further take $\phi(x) = \nabla f_D(x)$, this lemma provides the stability of the primal-dual gradient dynamics associated with the following optimization problem:

$$\begin{aligned} \min_{x \in \mathbb{R}^n} f_D(x) \\ \text{s.t. } S^\top x = \mathbf{0}. \end{aligned}$$

We then give a lemma which provides us a distributed optimal signal generator for (5).

Lemma 3: Suppose Assumptions 2 and 3 hold. Then the optimal consensus problem of (5) with $\dot{z}_i = v_i$, $y_i = z_i$ is solved by the following control:

$$\begin{cases} v_i = -\nabla f_i(z_i) - \sum_{j=1}^N a_{ij}(\lambda_i - \lambda_j) \\ \dot{\lambda}_i = \sum_{j=1}^N a_{ij}(z_i - z_j). \end{cases} \quad (16)$$

Furthermore, if Assumption 4 is satisfied, this control makes $y_i(t)$ approach the optimal solution y^* exponentially as $t \rightarrow \infty$ for $i = 1, \dots, N$.

Proof: The closed-loop system can be rewritten as follows:

$$\begin{cases} \dot{z} = -\nabla f(z) - (L \otimes I_m) \lambda \\ \dot{\lambda} = (L \otimes I_m) z \end{cases}$$

where $z = \text{col}(z_1, \dots, z_N)$, $\lambda = \text{col}(\lambda_1, \dots, \lambda_N)$ and $f(z)$ is determined by $f_1(z_1), \dots, f_N(z_N)$.

Its equilibrium point satisfies $-\nabla f(z^*) - (L \otimes I_m) \lambda^* = \mathbf{0}_N$ and $(L \otimes I_m) z^* = \mathbf{0}_{n \times m}$. As a result, there exists a θ such that $z_1^* = \dots = z_N^* = \theta$ since the null space of L is spanned by $\mathbf{1}_N$ under Assumption 2. Then one can obtain $\sum_{i=1}^N \nabla f_i(\theta) = \mathbf{0}$, which implies that θ is an optimal solution of (5). Thus, if we can prove the asymptotic stability of (16) at its equilibrium point (z^*, λ^*) , this algorithm indeed solves the distributed optimization problem determined by (5).

For this purpose, we take $\bar{z} = z - z^*$, $\bar{\lambda} = \lambda - \lambda^*$, and $\hat{\lambda}_1 = (r^\top \otimes I_m) \bar{\lambda}$, $\hat{\lambda}_2 = (R^\top \otimes I_m) \bar{\lambda}$. Since $(r^\top \otimes I_m) \bar{\lambda} = \mathbf{0}_m$, it follows that $\hat{\lambda}_1 \equiv \mathbf{0}_m$ and:

$$\begin{cases} \dot{\bar{z}} = -\mathbf{h} - (LR \otimes I_m) \hat{\lambda}_2 \\ \dot{\hat{\lambda}}_2 = (R^\top L \otimes I_m) \bar{z} \end{cases}$$

where $\mathbf{h} = \nabla f(z) - \nabla f(z^*)$. Denote $S = LR \otimes I_m$, which has a full-column rank since $R^\top L$ has a full-row rank. Hence, the above error system is of the form (12). Note that the strict convexity of $f_i(\cdot)$ implies $\bar{z}^\top \mathbf{h} > 0$ when $\bar{z} \neq \mathbf{0}$. By Lemma 2, one can obtain the asymptotic stability of this error system and hence the solvability of this optimal consensus problem under (16).

When $f_i(\cdot)$ satisfies Assumption 4, it follows that $\theta = y^*$ and \mathbf{h} is ϑ -Lipschitz with $\bar{z}^\top \mathbf{h} \geq \omega \bar{z}^\top \bar{z}$. Then the origin of the error

system is globally exponentially stable by Lemma 2, which implies that the proposed algorithm makes $y_i(t)$ converge to θ exponentially as $t \rightarrow \infty$ for $i = 1, \dots, N$. ■

Remark 4: As we mentioned, to design an optimal signal generator is to propose a proper algorithm which can asymptotically reproduce the optimal solution of (11) (for the single integrator). In other words, the practical dynamics of high-order plants are not involved in its design. In the distributed design, it means to find distributed algorithms to solve the optimal consensus problem for a group of single integrators. In fact, the generator (16) is inherently based on a primal-dual method to achieve an optimal consensus for single integrators.

From Lemma 3, one can find system (16) is an optimal signal generator for distributed optimization problem (5) if $f_i(\cdot)$ is strictly convex. Furthermore, under Assumption 4, the proposed generator is of exponential stability. By [40, Lemma 4.6], this optimal signal generator is robust to additive perturbations on its righthand side in the sense of input-to-state stability, which will play a role in our following design.

With the designed precompensator and optimal signal generator, we will complete the control loop by proposing proper reference-tracking controllers in the next section. In fact, this controller bridges the gap between the system (10a) in a normal form and the optimal signal generator (16), and plays an important role as an interface in the design of embedded systems [42]. Since z_i is time-varying, set-point regulators might fail to achieve our goal. To handle this issue, we adopt some system composition techniques and a high-gain strategy in its design.

IV. OPTIMAL OUTPUT CONSENSUS OF HIGH-ORDER AGENTS

In this section, we complete the embedded control design and prove that the proposed control algorithms can solve the optimal output consensus problem (5) for (3) in two gradient cases.

We start with a simple case (i.e., case I) when the gradient function $\nabla f_i(\cdot)$ is known to agent i , and then extend it to the second case (i.e., case II) when only the real-time gradient $\nabla f_i(y_i)$ is available in the design. Both state and output feedback designs are proposed for two cases in the following two respective sections.

Before the two sections, we give the following lemma for convergence analysis in the study of the two cases.

Lemma 4: Consider a cascaded nonlinear system as follows:

$$\dot{x}_1 = f_1(x_1) + g(x_1, x_2), \quad \dot{x}_2 = f_2(x_2). \quad (17)$$

Suppose for $i = 1, 2$:

- 1) $\dot{x}_1 = f_1(x_1)$ is globally exponentially stable at $x_1 = \mathbf{0}$;
- 2) $\dot{x}_2 = f_2(x_2)$ is asymptotically (or globally exponentially) stable at $x_2 = \mathbf{0}$;
- 3) $\|g(x_1, x_2)\| \leq M\|x_2\|$ for all x_1, x_2 with a constant $M > 0$.

Then the system (17) is also asymptotically (or globally exponentially) stable at the origin.

Proof: The asymptotic stability is a direct consequence of [40, Lemmas 4.6 and 4.7] and thus omitted here. We only prove the global exponential stability part. From the first two conditions and by [40, Th. 4.14], there exist a continuously differentiable function $V_i(x)$ and strictly positive constants c_{i1}, \dots, c_{i4} such that, for all x_i

$$c_{i1}\|x_i\|^2 \leq V_i(x_i) \leq c_{i2}\|x_i\|^2 \\ \dot{V}_i \leq -c_{i3}\|x_i\|^2, \quad \left\| \frac{\partial V_i}{\partial x_i} \right\| \leq c_{i4}\|x_i\|.$$

Denoting $x \triangleq \text{col}(x_1, x_2)$ and taking a Lyapunov function for the cascaded system (17) as $V(x) = V_1(x_1) + cV_2(x_2)$ with $c > 0$ to be determined later. Its derivative along (17) satisfies

$$\dot{V} = \frac{\partial V_1}{\partial x_1}(f_1(x_1) + g(x_1, x_2)) + c \frac{\partial V_2}{\partial x_2} f_2(x_2) \\ = -c_{13}\|x_1\|^2 + Mc_{14}\|x_1\|\|x_2\| - cc_{23}\|x_2\|^2.$$

By Young's inequality and letting $c > [(Mc_{14}^2 + 2c_{13})/(2c_{13}c_{23})]$, we have

$$\dot{V} \leq -\frac{c_{13}}{2}\|x_1\|^2 - \|x_2\|^2.$$

Since $\min\{c_{11}, cc_{21}\}\|x\|^2 \leq V(x_1, x_2) \leq \max\{c_{12}, cc_{22}\}\|x\|^2$, we invoke [40, Th. 4.10] and conclude the global exponential stability of (17). Thus, the conclusion follows. ■

A. Case I

In this case, the optimal signal generator can be independently implemented and asymptotically reproduce y^* by Lemma 3. Thus, we only need to propose tracking controllers for agents.

We first show our result on the high-order integrators for simplicity.

Theorem 1: Suppose Assumptions 2 and 3 hold. Then the optimal consensus problem (5) with agents of the form $x_i^{(n)} = v_i$, $y_i = x_i$ is solved by the following algorithm for any $\varepsilon > 0$:

$$\begin{cases} v_i = -\frac{1}{\varepsilon^n} \left[c_0(x_i - z_i) + \sum_{k=1}^{n-1} \varepsilon^k c_k x_i^{(k)} \right] \\ \dot{z}_i = -\nabla f_i(z_i) - \sum_{j=1}^N a_{ij}(\lambda_i - \lambda_j) \\ \dot{\lambda}_i = \sum_{j=1}^N a_{ij}(z_i - z_j) \end{cases} \quad (18)$$

where c_0, \dots, c_{n-1} are constants such that $\sum_{k=0}^{n-1} c_k s^k + s^n$ is Hurwitz. Furthermore, if Assumption 4 is satisfied, then $y_i(t)$ converges to y^* exponentially as $t \rightarrow \infty$ for $i = 1, \dots, N$.

Proof: Let $\hat{x}_i = \text{col}(x_i - z_i, \varepsilon x_i^{(1)}, \dots, \varepsilon^{n-1} x_i^{(n-1)})$, it follows:

$$\dot{\hat{x}}_i = \frac{1}{\varepsilon} \left(\hat{A} \otimes I_m \right) \hat{x}_i + (b \otimes I_m) \dot{z}_i \quad (19)$$

where $\hat{A} = \begin{bmatrix} 0 & I_{n-1} \\ -c_0 & [-c_1, \dots, -c_{n-1}] \end{bmatrix}$ and

$b = \text{col}(-1, \mathbf{0}_{n-1})$.

Then the closed-loop system is

$$\begin{cases} \dot{\hat{x}}_i = \frac{1}{\varepsilon} A_m \hat{x}_i + b_m \left[-\nabla f_i(z_i) - \sum_{j=1}^N a_{ij}(\lambda_i - \lambda_j) \right] \\ \dot{z}_i = -\nabla f_i(z_i) - \sum_{j=1}^N a_{ij}(\lambda_i - \lambda_j) \\ \dot{\lambda}_i = \sum_{j=1}^N a_{ij}(z_i - z_j) \end{cases} \quad (20)$$

where $A_m = \hat{A} \otimes I_m$ and $b_m = b \otimes I_m$.

Due to the choice of c_0, \dots, c_{n-1} , the matrix A_m is Hurwitz, which implies $\|e^{(t-t_0)(1/\varepsilon)A_m}\| \leq k_1 e^{-(k_2/\varepsilon)(t-t_0)}$ for some positive constants k_1 and k_2 . Solving (19) and using this bound, we obtain

$$\|\hat{x}(t)\| \leq k_1 e^{-\frac{k_2}{\varepsilon}(t-t_0)} \|\hat{x}(t_0)\| + \frac{k_1 \varepsilon \|b_m\|}{k_2} \sup_{t_0 \leq \tau \leq t} \|\dot{z}(\tau)\|.$$

By Lemma 3, z_i converges to y^* and \dot{z}_i converges to $\mathbf{0}$ when $t \rightarrow \infty$ for $i = 1, \dots, N$. For any given $\delta > 0$, there is a time $T_1 > 0$ such that $\|\dot{z}(t)\| \leq [(\delta k_2)/(2k_1 \varepsilon \|b_m\|)]$ for all $t > T_1$. Choose $t_0 > T_1$ and there exists a $T_2 > 0$ such that $k_1 e^{-(k_2/\varepsilon)(t-t_0)} \|x(t_0)\| \leq (\delta/2)$ for $t > T_2$. Thus, $\|\hat{x}(t)\| \leq \delta$ for all $t > \max\{T_1, T_2\}$, which leads to the convergence of $x_i - z_i$ for $i = 1, \dots, N$. Recalling $\lim_{t \rightarrow +\infty} z_i = y^*$ gives the convergence of x_i with respect to y^* for $i = 1, \dots, N$.

To prove the exponential convergence, we rewrite the whole closed-loop system as follows:

$$\begin{cases} \dot{\hat{x}}_i = \frac{1}{\varepsilon} A_m \hat{x}_i + b_m \dot{z}_i \\ \dot{z}_i = -\mathbf{h}_i - \sum_{j=1}^N a_{ij} (\bar{\lambda}_i - \bar{\lambda}_j) \\ \dot{\bar{\lambda}}_i = \sum_{j=1}^N a_{ij} (\bar{z}_i - \bar{z}_j) \end{cases} \quad (21)$$

where $\hat{x}_i = \text{col}(x_i - z_i, \varepsilon x_i^{(1)}, \dots, \varepsilon^{n-1} x_i^{(n-1)})$, $\bar{z}_i = z_i - y^*$, $\bar{\lambda}_i = \lambda_i - \lambda_i^*$, and $\mathbf{h}_i = \nabla f_i(z_i) - \nabla f_i(y^*)$ for $i = 1, \dots, N$.

If Assumption 4 holds, (z_i, λ_i) exponentially converges to its equilibrium point (y^*, λ_i^*) as $t \rightarrow \infty$ for $i = 1, \dots, N$ by Lemma 3. From the Lipschitzness of $\nabla f_i(\cdot)$, the cross-term \dot{z}_i is also Lipschitz with respect to $(\bar{z}_i, \bar{\lambda}_i)$. By Lemma 4, the exponential stability of (21) is achieved, which implies the conclusion. ■

Remark 5: Clearly, as the gradients of local cost functions are analytically known, the optimal signal generator can be implemented independently. Then the whole system is in a cascading form and the strict convexity will suffice the solvability of this optimal output consensus problem. When Assumption 4 is satisfied, the convergence can be exponentially fast by Lemma 4.

Based on Theorem 1 and the normal form (10a), we go back to investigate linear minimum-phase agents of the form (3) and get the following controller:

$$\begin{cases} \tilde{u}_i = \tilde{R}^{-1} (-\Upsilon \chi_i^a - S \chi_i^b + u_i) \\ v_i = -\frac{1}{\varepsilon^n} \left[c_0 (y_i - z_i) + \sum_{k=1}^{n-1} \varepsilon^k c_k y_i^{(k)} \right] \\ \dot{z}_i = -\nabla f_i(z_i) - \sum_{j=1}^N a_{ij} (\lambda_i - \lambda_j) \\ \dot{\lambda}_i = \sum_{j=1}^N a_{ij} (z_i - z_j) \end{cases} \quad (22)$$

where c_1, \dots, c_{n-1} are selected as given in Theorem 1, and the relationship between u_i and v_i is described by (9).

Theorem 2: Suppose Assumptions 1–3 hold. Then the optimal output consensus problem of system (3) is solved by (22) for any $\varepsilon > 0$. Furthermore, if Assumption 4 is satisfied, $y_i(t)$ converges to y^* exponentially as $t \rightarrow \infty$ for $i = 1, \dots, N$.

Proof: Based on the above analysis, the output variables of system (10a) under control v_i and system (3) under control (22) are the same. Thus, we only have to investigate the convergence of (10a) under the above control. Clearly, the subsystem (10b) and (10c) is equivalent to the n th-order integrator. Let $\tilde{\chi}_i^a = \chi_i^a - \chi_i^{a*}$, $\tilde{\chi}_i^b = \chi_i^b - \chi_i^{b*}$ [where χ_i^{a*}, χ_i^{b*}

are the equilibrium point of (10a) and (10b) determined by y^*] and define $\hat{\chi}_i^c$ as \hat{x}_i in Theorem 1. The whole systems is rewritten as

$$\begin{cases} \dot{\tilde{\chi}}_i^a = \Pi \tilde{\chi}_i^a + \Psi \tilde{\chi}_i^b \\ \dot{\hat{\chi}}_i^c = \frac{1}{\varepsilon} A_m \hat{\chi}_i^c + b_m \dot{z}_i \\ \dot{z}_i = -\mathbf{h}_i - \sum_{j=1}^N a_{ij} (\bar{\lambda}_i - \bar{\lambda}_j) \\ \dot{\bar{\lambda}}_i = \sum_{j=1}^N a_{ij} (\bar{z}_i - \bar{z}_j). \end{cases}$$

Apparently, it is in a cascading form with the first subsystem as a driven one. Recalling the minimum-phase property of (3), the matrix Π is Hurwitz. According to Theorem 1, the output y_i of the last three subsystems asymptotically converges to the optimal solution y^* , then the asymptotic convergence of the whole cascaded system can be obtained by Lemma 4. When Assumption 4 holds, the global exponential stability can be obtained in a similar way. The proof is thus complete. ■

Next, let us consider the case when only the output variables of each agent can be obtained because it may be difficult to get or measure all the state variables in some situations. Since the optimal signal generator is independently implemented, we only have to focus on the tracking part. To solve the problem, we consider an output feedback version of the proposed high-gain embedded control by proposing an observer-based output feedback design.

Since system (3) is minimum phase and therefore detectable, we can design the following local observer for agent i ($i = 1, \dots, N$) of the form (3):

$$\begin{cases} \dot{\hat{y}}_i = \hat{y}_i^{(1)} - l_1 (\hat{y}_i - y_i) \\ \dot{\hat{y}}_i^{(1)} = \hat{y}_i^{(2)} - l_2 (\hat{y}_i - y_i) \\ \dots \\ \dot{\hat{y}}_i^{(n-1)} = v_i - l_n (\hat{y}_i - y_i) \\ \dot{\hat{\chi}}_i^a = \Pi \hat{\chi}_i^a + \Psi \hat{\chi}_i^b \end{cases} \quad (23)$$

where $\hat{\chi}_i^a, \hat{\chi}_i^b, \hat{y}_i$, and $\hat{y}_i^{(\iota)}$ are the estimations of χ_i^a, χ_i^b, y_i , and $y_i^{(\iota)}$, respectively, with $\iota \in \{1, \dots, n-1\}$, and l_1, \dots, l_n are constants such that $p_l(s) = s^n + l_1 s^{n-1} + \dots + l_{n-1} s + l_n$ is Hurwitz. Then we substitute these variables by their estimations and propose the following distributed control for system (3):

$$\begin{cases} \tilde{u}_i = \tilde{R}^{-1} (-\Upsilon \hat{\chi}_i^a - S \hat{\chi}_i^b + u_i) \\ v_i = -\frac{1}{\varepsilon^n} \left[c_0 (y_i - z_i) + \sum_{k=1}^{n-1} \varepsilon^k c_k \hat{y}_i^{(k)} \right] \\ \dot{z}_i = -\nabla f_i(z_i) - \sum_{j=1}^N a_{ij} (\lambda_i - \lambda_j) \\ \dot{\lambda}_i = \sum_{j=1}^N a_{ij} (z_i - z_j) \end{cases} \quad (24)$$

where c_0, \dots, c_{n-1} and $\varepsilon > 0$ are defined as before, the relationship between u_i and v_i is described by (9).

We then give the following results for the output feedback design.

Theorem 3: Suppose Assumptions 1–3 hold. Then the optimal output consensus problem for the agents of the form (3) is solved by (24) for any $\varepsilon > 0$. Furthermore, if Assumption 4 is satisfied, $y_i(t)$ converges to y^* exponentially as $t \rightarrow \infty$ for $i = 1, \dots, N$.

Proof: Taking $\bar{e}_i = \text{col}(y_i - \hat{y}_i, \dots, y_i^{(n)} - \hat{y}_i^{(n)})$ and $\bar{\chi}_i^a = \hat{\chi}_i^a - \chi_i^a, \bar{\chi}_i^b = \hat{\chi}_i^b - \chi_i^b$ gives

$$\begin{cases} \dot{\bar{e}}_i = (A_l \otimes I_m) \bar{e}_i \\ \dot{\bar{\chi}}_i^a = \Pi \bar{\chi}_i^a + \Psi \bar{\chi}_i^b \end{cases}$$

where

$$A_l = \begin{bmatrix} -\mathbf{1}_{n-1} & I_{n-1} \\ -l_n & \mathbf{0}_{n-1}^\top \end{bmatrix}, \quad \mathbf{l}_{n-1} = \text{col}(l_1, \dots, l_{n-1}).$$

Recalling the Hurwitzness of Π , we apply Lemma 4 and obtain $\hat{\chi}_i^a, \hat{\chi}_i^b, \hat{y}_i, \hat{y}_i^{(l)}$ exponentially converge to χ_i^a, χ_i^b, y_i and $y_i^{(l)}$ with $l \in \{1, \dots, n-1\}$ for $i \in \{1, 2, \dots, N\}$, respectively.

Since substituting χ_i^a, χ_i^b, y_i and $y_i^{(l)}$ by their estimations will not change the equilibrium point, we only have to prove the stability of the new system with respect to its equilibrium point under observer-based control laws. For this purpose, we repeat the whole error systems as follows:

$$\begin{cases} \dot{\tilde{\chi}}_i^a = \Pi \tilde{\chi}_i^a + \Psi \tilde{\chi}_i^b \\ \dot{\tilde{\chi}}_i^c = \frac{1}{\varepsilon} A_m \tilde{\chi}_i^c + b_m \dot{\tilde{z}}_i + \Gamma_i \\ \dot{\tilde{z}}_i = -\mathbf{h}_i - \sum_{j=1}^N a_{ij} (\bar{\lambda}_i - \bar{\lambda}_j) \\ \dot{\tilde{\lambda}}_i = \sum_{j=1}^N a_{ij} (\bar{z}_i - \bar{z}_j) \\ \dot{\tilde{\chi}}_i^a = \Pi \tilde{\chi}_i^a + \Psi \tilde{\chi}_i^b \\ \dot{\tilde{e}}_i = (A_l \otimes I_m) \tilde{e}_i \end{cases}$$

where Γ_i is Lipschitz in $(\bar{\chi}_i^a, \bar{e}_i)$ determined by (7) and (24).

This whole system is again in a cascaded form with the last two subsystems as the driving one. By Theorem 2 and Lemma 4, this implies the asymptotic stability when $f_i(\cdot)$ is strictly convex and global exponential stability under Assumption 4. Hence, the proof is complete. ■

Remark 6: Based on the above analysis, it is worthwhile to mention that we can solve the problem by making the optimization design and tracking control design of high-order dynamics almost independent, because the generator for the optimization, independent of the high-order dynamics, is “separately” designed by each agent, and then embedded into the designed reference-tracking control of the high-order dynamics. As a result, the design complexities brought by high-order dynamics are decoupled from those by the optimization task. By solving the two simpler subproblems, the optimal output consensus problem can be solved via constructive controllers. Also, this embedded control framework enjoys a large flexibility in choosing optimal signal generators and tracking controllers, and therefore, may be useful in various optimization algorithms.

B. Case II

In some cases, the function $\nabla f_i(\cdot)$ is hard to obtain and only the real-time gradient $\nabla f_i(y_i)$ is available, and then the optimal signal generator cannot be independently implemented. Suppose we substitute $\nabla f_i(z_i)$ by $\nabla f_i(y_i)$ in (16), there will be a mismatching error $\nabla f_i(z_i) - \nabla f_i(y_i)$ in the control. As stated in Lemma 3, the proposed optimal signal generator under Assumption 4 is actually robust with respect to additive perturbations. Thus, these mismatching errors can be handled and compensated by its robustness along with small-gain techniques and a high-gain design.

We first show the result on high-order integrators for simplicity.

Theorem 4: Suppose Assumptions 1, 2, and 4 hold. Then there exists a constant $\varepsilon^* > 0$ such that the optimal output consensus problem (5) with agents of the form $x_i^{(n)} = v_i, y_i =$

x_i is solved by the following control algorithm for any $\varepsilon \in (0, \varepsilon^*)$:

$$\begin{cases} v_i = -\frac{1}{\varepsilon^n} \left[c_0(x_i - z_i) + \sum_{k=1}^{n-1} \varepsilon^k c_k x_i^{(k)} \right] \\ \dot{z}_i = -\nabla f_i(y_i) - \sum_{j=1}^N a_{ij} (\lambda_i - \lambda_j) \\ \dot{\lambda}_i = \sum_{j=1}^N a_{ij} (z_i - z_j) \end{cases} \quad (25)$$

where c_0, \dots, c_{n-1} are constants such that $\sum_{k=0}^{n-1} c_k s^k + s^n$ is Hurwitz. Moreover, $y_i(t)$ converges to the optimal solution y^* exponentially as $t \rightarrow \infty$ for $i = 1, \dots, N$.

Proof: Letting $\bar{z}_i = z_i - y^*$ and $\bar{\lambda}_i = \lambda_i - \lambda_i^*$ gives

$$\begin{cases} \dot{\hat{\chi}}_i = \frac{1}{\varepsilon} A_m \hat{\chi}_i + b_m \left[-\mathbf{h}_i - \Delta_i - \sum_{j=1}^N a_{ij} (\bar{\lambda}_i - \bar{\lambda}_j) \right] \\ \dot{\bar{z}}_i = -\mathbf{h}_i - \Delta_i - \sum_{j=1}^N a_{ij} (\bar{\lambda}_i - \bar{\lambda}_j) \\ \dot{\bar{\lambda}}_i = \sum_{j=1}^N a_{ij} (\bar{z}_i - \bar{z}_j) \end{cases}$$

where $\mathbf{h}_i = \nabla f_i(z_i) - \nabla f_i(y^*)$ and $\Delta_i = \nabla f_i(y_i) - \nabla f_i(z_i)$, or in a compact form

$$\begin{cases} \dot{\hat{\chi}} = \frac{1}{\varepsilon} (I_N \otimes A_m) \hat{\chi} + (I_N \times b_m) \left[-\mathbf{h} - \Delta - (L \otimes I_m) \bar{\lambda} \right] \\ \dot{\bar{z}} = -\mathbf{h} - \Delta - (L \otimes I_m) \bar{\lambda} \\ \dot{\bar{\lambda}} = (L \otimes I_m) \bar{z} \end{cases} \quad (26)$$

where $\hat{\chi} = \text{col}(\hat{\chi}_1, \dots, \hat{\chi}_N) \in \mathbb{R}^{nN}$, \mathbf{h} and Δ are determined by \mathbf{h}_i and Δ_i .

Perform another coordinate transformation: $\hat{\lambda}_1 = (r^\top \otimes I_m) \bar{\lambda}, \hat{\lambda}_2 = (R^\top \otimes I_m) \bar{\lambda}$. Since $(r^\top \otimes I_m) \dot{\bar{\lambda}} = \mathbf{0}$, it follows that $\dot{\hat{\lambda}}_1 \equiv \mathbf{0}$ and:

$$\begin{cases} \dot{\hat{\chi}} = \frac{1}{\varepsilon} (I_N \otimes A_m) \hat{\chi} + (I_N \times b_m) \bar{\Gamma} \\ \dot{\bar{z}} = -\mathbf{h} - \Delta - (LR \otimes I_m) \hat{\lambda}_2 \\ \dot{\hat{\lambda}}_2 = (R^\top L \otimes I_m) \bar{z} \end{cases} \quad (27)$$

where $\bar{\Gamma} = -\mathbf{h} - \Delta - (LR \otimes I_m) \hat{\lambda}_2$.

Let $\hat{z} \triangleq \text{col}(\bar{z}, \hat{\lambda}_2)$. It can be easily verified that Δ is ϑ -Lipschitz in $x - z$ and hence $\hat{\chi}$. Note that $\bar{\Gamma}$ is also $\bar{\vartheta}_1$ -Lipschitz in $\hat{\lambda}_2$ for a positive constant $\bar{\vartheta}_1$. Thus, there exists positive constants $\bar{\vartheta}_2$ and $\bar{\vartheta}_3$ such that $\|\bar{\Gamma}\| \leq \bar{\vartheta}_2 \|\hat{\chi}\| + \bar{\vartheta}_3 \|\hat{z}\|$.

Next, we invoke a small-gain technique on (27) to prove this theorem via tuning ε . By Lemma 2 or the proof of Lemma 3, the following (nominal) system

$$\begin{cases} \dot{\bar{z}} = -\mathbf{h} - (LR \otimes I_m) \hat{\lambda}_2 \\ \dot{\hat{\lambda}}_2 = (R^\top L \otimes I_m) \bar{z} \end{cases} \quad (28)$$

is globally exponentially stable under the given assumptions.

Recalling the Lipschitzness of \mathbf{h} in \bar{z} , we then apply the converse Lyapunov theorem [40, Th. 4.15] to this system, that is, there is a continuously differentiable Lyapunov function $V_1(\cdot)$ such that

$$\begin{aligned} \hat{c}_1 \|\hat{z}\|^2 &\leq V_1(\hat{z}) \leq \hat{c}_2 \|\hat{z}\|^2 \\ \frac{\partial V_1}{\partial \bar{z}} \left[-\mathbf{h} - (LR \otimes I_m) \hat{\lambda}_2 \right] + \frac{\partial V_1}{\partial \hat{\lambda}_2} \left[(R^\top L \otimes I_m) \bar{z} \right] &\leq -\hat{c}_3 \|\hat{z}\|^2 \\ \left\| \frac{\partial V_1}{\partial \hat{z}} \right\| &\leq \hat{c}_4 \|\hat{z}\| \end{aligned}$$

for some positive constants $\hat{c}_1, \hat{c}_2, \hat{c}_3$, and \hat{c}_4 .

Since A_m is Hurwitz, there is a positive definite matrix $P_m \in \mathbb{R}^{nm \times nm}$ satisfying $A_m^\top P_m + P_m A_m = -I_{nm}$. We take a quadratic

Lyapunov function $V = \hat{x}^\top (I_N \otimes P_m) \hat{x} + V_1(\hat{z})$, whose derivative along the trajectory of system (27) is

$$\begin{aligned} \dot{V} &= 2\hat{x}^\top (I_N \otimes P_m) \left[\frac{1}{\varepsilon} (I_N \otimes A_m) \hat{x} + (I_N \times b_m) \bar{\Gamma} - \frac{\partial V_1}{\partial \hat{z}} \dot{\hat{z}} \right] \\ &\leq \frac{1}{\varepsilon} \hat{x}^\top \left[I_N \otimes (P_m A_m + A_m^\top P_m) \right] \hat{x} + 2\hat{x}^\top (I_N \otimes P_m b_m) \bar{\Gamma} \\ &\quad + \frac{\partial V_1}{\partial \bar{z}} \left[-\mathbf{h} - (LR \otimes I_m) \hat{\lambda}_2 \right] + \frac{\partial V_1}{\partial \bar{\lambda}_2} \left(R^\top L \otimes I_m \right) z \\ &\quad - \frac{\partial V_1}{\partial \bar{z}} \Delta \\ &\leq -\frac{1}{\varepsilon} \|\hat{x}\|^2 + \hat{\vartheta}_2 \|\hat{x}\|^2 + \hat{\vartheta}_3 \|\hat{x}\| \|\hat{z}\| - \hat{c}_3 \|\hat{z}\|^2 + \hat{c}_4 \vartheta \|\hat{x}\| \|\bar{z}\| \\ &\leq -\left(\frac{1}{\varepsilon} - \hat{\vartheta}_2 \right) \|\hat{x}\|^2 + \hat{\vartheta}_4 \|\hat{x}\| \|\hat{z}\| - \hat{c}_3 \|\hat{z}\|^2 \end{aligned}$$

where $\hat{\vartheta}_2 = 2\bar{\vartheta}_2 \|(I_N \otimes P_m b_m)\|$, $\hat{\vartheta}_3 = 2\bar{\vartheta}_3 \|(I_N \otimes P_m b_m)\|$, and $\hat{\vartheta}_4 = \hat{\vartheta}_3 + \hat{c}_4 \vartheta$.

By the Young's inequality, we have

$$\begin{aligned} \dot{V} &\leq -\left(\frac{1}{\varepsilon} - \hat{\vartheta}_2 \right) \|\hat{x}\|^2 + \frac{\hat{\vartheta}_4^2}{2\hat{c}_3} \|\hat{x}\|^2 + \frac{\hat{c}_3}{2} \|\hat{z}\|^2 - \hat{c}_3 \|\hat{z}\|^2 \\ &= -\left(\frac{1}{\varepsilon} - \hat{\vartheta}_2 - \frac{\hat{\vartheta}_4^2}{2\hat{c}_3} \right) \|\hat{x}\|^2 - \frac{\hat{c}_3}{2} \|\hat{z}\|^2. \end{aligned}$$

We take

$$\varepsilon^* = \frac{2\hat{c}_3}{2\hat{c}_3\hat{\vartheta}_2 + \hat{\vartheta}_4^2} \quad (29)$$

and obtain that for any $\varepsilon \in (0, \varepsilon^*)$

$$\dot{V} \leq -\left(\frac{1}{\varepsilon} - \frac{1}{\varepsilon^*} \right) \|\hat{x}\|^2 - \frac{\hat{c}_3}{2} \|\hat{z}\|^2$$

which implies the exponential stability of (27) and therefore, the exponential convergence of x_i with respect to y^* as $t \rightarrow \infty$. Thus, the proof is complete. ■

Based on Theorem 4 and the normal form (10a), we go back to investigate linear minimum-phase agents of the form (3) and the following controller is readily obtained, that is:

$$\begin{cases} \tilde{u}_i = \tilde{R}^{-1} (-\Upsilon \chi_i^a - S \chi_i^b + u_i) \\ v_i = -\frac{1}{\varepsilon^n} \left[c_0 (y_i - z_i) + \sum_{k=1}^{n-1} \varepsilon^k c_k y_i^{(k)} \right] \\ \dot{z}_i = -\nabla f_i(y_i) - \sum_{j=1}^N a_{ij} (\lambda_i - \lambda_j) \\ \dot{\lambda}_i = \sum_{j=1}^N a_{ij} (z_i - z_j) \end{cases} \quad (30)$$

where c_1, \dots, c_{n-1} are selected as that in Theorem 4, and the relationship between u_i and v_i is described by (9).

Here is our main result of (30) on general linear systems based on the embedded control scheme.

Theorem 5: Suppose Assumptions 1, 2, and 4 hold. Then there is a constant $\varepsilon^* > 0$ such that the optimal output consensus problem (5) with agents of the form (3) is solved by the control (30) for any $\varepsilon \in (0, \varepsilon^*)$. Moreover, $y_i(t)$ converges to y^* exponentially as $t \rightarrow \infty$ for $i = 1, \dots, N$.

Proof: Following similar arguments in the proof of Theorem 2. Clearly, the subsystem (10b) and (10c) is equivalent to the n th-order integrator. Let $\tilde{\chi}_i^a = \chi_i^a - \chi_i^{a^*}$, $\tilde{\chi}_i^b = \chi_i^b - \chi_i^{b^*}$ [where $(\chi_i^{a^*}, \chi_i^{b^*})$ is the equilibrium

point of (10a) and (10b) determined by y^*] and take a similar coordinate transformation as that given in Theorem 4. Then the whole error system can be expressed in a cascaded form

$$\begin{cases} \dot{\tilde{\chi}}_i^a = \Pi \tilde{\chi}_i^a + \Psi \tilde{\chi}_i^b \\ \dot{\tilde{\chi}}_i^c = \frac{1}{\varepsilon} A_m \tilde{\chi}_i^c + b_m \left[-\mathbf{h}_i - \Delta_i - \sum_{j=1}^N a_{ij} (\bar{\lambda}_i - \bar{\lambda}_j) \right] \\ \dot{\tilde{z}}_i = -\mathbf{h}_i - \Delta_i - \sum_{j=1}^N a_{ij} (\bar{\lambda}_i - \bar{\lambda}_j) \\ \dot{\tilde{\lambda}}_i = \sum_{j=1}^N a_{ij} (\bar{z}_i - \bar{z}_j). \end{cases}$$

Note that the matrix Π is Hurwitz and $\Psi \tilde{\chi}_i^b$ is Lipschitz with respect to $(\tilde{\chi}_i^c, \bar{z}_i)$. Set ε^* as defined in (29) and by the proof of Theorem 4, the subsystem $(\tilde{\chi}_i^c, \bar{z}_i, \bar{\lambda}_i)$ is then globally exponentially stable with respect to its equilibrium point under control input v_i for any $0 < \varepsilon < \varepsilon^*$. Global exponential stability of the above cascaded system can be obtained by Lemma 4, which implies that the state trajectory of this agent is bounded and its output y_i converges to y^* exponentially as $t \rightarrow \infty$. The proof is thus complete. ■

It can be found that for the given multiagent system, the tracking control of each agent can be achieved by tuning only one input parameter ε . For different optimization problems, we can simply replace the local cost function $f_i(\cdot)$ in the optimal signal generator and then adjust the gain parameter ε , which transforms a controller design problem into a parameter-tuning one. Therefore, the embedded control scheme is much simpler than redesigning a complete new algorithm for these high-order agents, which may be favorable in large-scale networks.

Remark 7: Similar problems have been investigated for the first-order system [17], [18] with vector relative degree $(1, \dots, 1)$ and the second-order system [27], [28] with vector relative degree $(2, \dots, 2)$. It is remarkable that the optimal signal generator used here actually has decoupled the complexity of optimization task from that of the high-order dynamics tracking problem, which may facilitate the design for even more complex agent dynamics.

Next, let us consider its output feedback version. As in case I, by attaching the high-gain observer (23), we have the following distributed control for system (3):

$$\begin{cases} \tilde{u}_i = \tilde{R}^{-1} (-\Upsilon \hat{\chi}_i^a - S \hat{\chi}_i^b + u_i) \\ v_i = -\frac{1}{\varepsilon^n} \left[c_0 (y_i - z_i) + \sum_{k=1}^{n-1} \varepsilon^k c_k \hat{y}_i^{(k)} \right] \\ \dot{z}_i = -\nabla f_i(y_i) - \sum_{j=1}^N a_{ij} (\lambda_i - \lambda_j) \\ \dot{\lambda}_i = \sum_{j=1}^N a_{ij} (z_i - z_j) \end{cases} \quad (31)$$

where c_0, \dots, c_{n-1} are defined as before and $\varepsilon > 0$ is to be determined later.

We then have the following theorem for this design.

Theorem 6: Suppose Assumptions 1, 2, and 4 hold. Then there is a constant $\varepsilon^* > 0$ such that the optimal output consensus problem (5) with agents of the form (3) is solved by (31) for any $\varepsilon \in (0, \varepsilon^*)$. Moreover, $y_i(t)$ converge to the optimal solution y^* exponentially as $t \rightarrow \infty$ for $i = 1, \dots, N$.

Proof: The proof is similar to Theorem 3. In fact, following the same arguments, we only have to prove the stability of the

whole error system:

$$\begin{cases} \dot{\tilde{\chi}}_i^a = \Pi \tilde{\chi}_i^a + \Psi \tilde{\chi}_i^b \\ \dot{\tilde{\chi}}_i^c = \frac{1}{\varepsilon} A_m \tilde{\chi}_i^c + b_m \left[-\mathbf{h}_i - \Delta_i - \sum_{j=1}^N a_{ij} (\tilde{\lambda}_i - \tilde{\lambda}_j) \right] + \Gamma_i \\ \dot{\tilde{z}}_i = -\mathbf{h}_i - \Delta_i - \sum_{j=1}^N a_{ij} (\tilde{\lambda}_i - \tilde{\lambda}_j) \\ \dot{\tilde{\lambda}}_i = \sum_{j=1}^N a_{ij} (\tilde{z}_i - \tilde{z}_j) \\ \dot{\tilde{\chi}}_i^a = \Pi \tilde{\chi}_i^a + \Psi \tilde{\chi}_i^b \\ \dot{\tilde{e}}_i = (A_l \otimes I_m) \tilde{e}_i \end{cases}$$

where Γ_i is Lipschitz in $(\tilde{\chi}_i^a, \tilde{e}_i)$ determined by (7) and (31).

This system is again in a cascaded form. Choosing ε^* as defined in (29), one can obtain its global exponential stability for any $\varepsilon \in (0, \varepsilon^*)$ under Assumption 4 by Lemma 4 and Theorem 5. Hence, the proof is complete. ■

Remark 8: In practice, bounded uncertainties may come from the communication noise and gradient calculation. Suppose Assumptions 1, 2, and 4 hold, it can be easily verified that our algorithm is robust in the sense of bounded-input bounded-output stability with respect to those uncertainties as its input and approximation error of $y_i - y_i^*$ as its output. In fact, we can make the approximation error arbitrarily small by adjusting the gain parameter ε .

Remark 9: As we remarked before, the optimal output consensus problem is a combination of the two hot topics: 1) output consensus of high-order agents [6]–[9] and 2) distributed optimization for single integrators [15]–[18]. Due to the couplings between the distributed optimization requirement and high-order dynamic processes, it is much more challenging than traditional distributed optimization or output consensus problem. This is why we come up with the optimal signal generator to decouple the optimization design from the dynamics in order to simplify the whole design complexities. Compared with the new result given in [29], this embedded scheme leads to a generally constructive way to solve the problem.

V. SIMULATIONS

In this section, we present two examples to illustrate our problem and the effectiveness of our designs.

Example 1: Consider an optimal rendezvous problem [8] of wheeled robots with the following dynamics:

$$\begin{cases} \dot{r}_i^x = v_i \cos(\theta_i) \\ \dot{r}_i^y = v_i \sin(\theta_i) \\ \dot{\theta}_i = \omega_i \\ \dot{v}_i = \frac{1}{m_i} F_i \\ \dot{\omega}_i = \frac{1}{J_i} \tau_i \end{cases}$$

where (r_i^x, r_i^y, θ_i) are the inertia center's position and orientation of the i th robot, (v_i, ω_i) the linear and angular speed, (F_i, τ_i) the applied force and torque, (m_i, J_i) the mass and moment of inertia for $i = 1, \dots, 5$. Let d_i represent the distance between the hand position and inertia center of the i th robot. Applying feedback linearization as in [8] about the hand position $\tilde{x}_i \triangleq (r_i^x + d_i \cos(\theta_i), r_i^y + d_i \sin(\theta_i))$ yields a simple linear dynamics as follows:

$$\dot{\tilde{x}}_i = \tilde{v}_i, \dot{\tilde{v}}_i = \tilde{u}_i, y_i = \tilde{x}_i.$$

It apparently satisfies Assumption 1 with relative degree (2, 2).

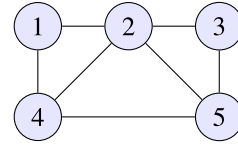


Fig. 2. Interaction topology of the multiagent system.

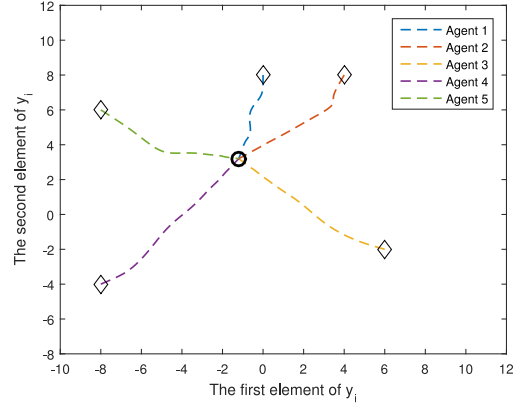


Fig. 3. Rendezvous of five mobile robots under state feedback control (18).

To drive all hands of robots to rendezvous at a common point that minimizes the aggregate distance from their starting points to this final location, the cost functions satisfying Assumption 4 are as $f_i(y) = (1/2)\|y - y_i(0)\|^2$ and $f(y) = (1/2) \sum_{i=1}^5 \|y - y_i(0)\|^2$ ($i = 1, \dots, 5$). We can easily check that the optimal solution of the global cost function is $\text{Aver}(y(0)) \triangleq (1/N) \sum_{i=1}^N y_i(0)$. The communication graph among these robots satisfying Assumption 2 is depicted as Fig. 2 with all the edge weights as 1. Recalling Theorem 1, the rendezvous problem can be solved by controller (18).

Because $\nabla f_i(y) = y - y_i(0)$, the generator (16) reduces to

$$\begin{cases} \dot{z}_i = -(z_i - y_i(0)) - \sum_{j=1}^N a_{ij} (\lambda_i - \lambda_j) \\ \dot{\lambda}_i = \sum_{j=1}^N a_{ij} (z_i - z_j). \end{cases}$$

Thus, the state feedback control (18) is as follows:

$$\begin{cases} \tilde{u}_i = -\frac{1}{\varepsilon^2} [c_0 (y_i - z_i) + \varepsilon c_1 \tilde{v}_i] \\ \dot{z}_i = -(z_i - y_i(0)) - \sum_{j=1}^N a_{ij} (\lambda_i - \lambda_j) \\ \dot{\lambda}_i = \sum_{j=1}^N a_{ij} (z_i - z_j). \end{cases}$$

Take $c_0 = 4, c_1 = 8, \varepsilon = 1$ and all initials (randomly) in $[-10, 10]^8$. The local minimizers, i.e., their initial points, are marked by diamonds, the global optimal solution y^* by a circle. The simulation result is given in Fig. 3 and all robots achieve the optimal rendezvous at y^* .

Example 2. To verify the effectiveness of our embedded design, we then consider a high-order multiagent system modified from [7] with more complex objective functions

$$\dot{\tilde{x}}_i = \tilde{A} \tilde{x}_i + \tilde{B} \tilde{u}_i, \quad y_i = \tilde{C} \tilde{x}_i, \quad i = 1, \dots, 5 \quad (32)$$

where

$$\tilde{A} = \begin{bmatrix} 0.1 & 0.1 & -2.5 & 0.5 \\ 0 & -0.4 & 0 & -1.5 \\ 2 & 0 & 0 & -1 \\ 0 & 0.2 & 1 & 0.8 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

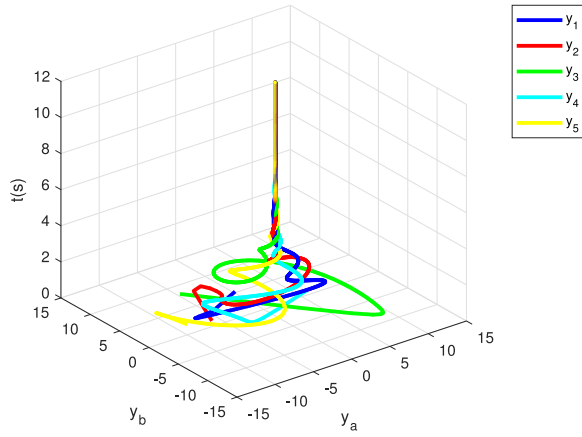


Fig. 4. Phase portraits of y_i under output feedback control (31).

$$\tilde{C} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The local cost functions are as follows:

$$f_1(y) = (y_a - 8)^2 + (y_b - 1)^2$$

$$f_2(y) = \frac{y_a^2}{20\sqrt{(y_a)^2 + 1}} + \frac{y_b^2}{20\sqrt{(y_b)^2 + 1}} + \|y\|^2$$

$$f_3(y) = \frac{(y_a)^2}{80 \ln((y_a)^2 + 2)} + \frac{(y_b)^2}{80 \ln((y_b)^2 + 2)} + \|y - 5 \times \mathbf{1}_2\|^2$$

$$f_4(y) = \ln(e^{-0.05y_a} + e^{0.05y_a}) + \ln(e^{-0.05y_b} + e^{0.05y_b}) + \|y\|^2$$

$$f_5(y) = \frac{y_a^2}{25\sqrt{(y_a)^2 + 1}} + \frac{y_b^2}{25\sqrt{(y_b)^2 + 1}} + \|y\|^2 + \mathbf{1}_2^\top y$$

where $y = [y_a \ y_b]^T$. The interaction topology of this multiagent system is still taken as Fig. 2. It can be verified that the vector relative degree of agents is (2, 1) and Assumptions 1–4 hold. Additionally, $y_a^* = 2.5$, $y_b^* = 1.1$ by minimizing the global cost function $f(y) = \sum_{i=1}^5 f_i^*(y)$.

Assume that agents only have real-time gradients. Recalling Theorem 6, this output optimal consensus problem can be solved by output feedback controller (31).

In fact, by taking $\phi_i = [1, 1, 0, -1]$, the system (32) can be put into (6) as follows:

$$\begin{cases} \dot{\chi}_i^a = -0.5\chi_i^a + [-3.5, 0.3, -2]\chi_i^b \\ \dot{\xi}_{11}^{ib} = \xi_{12}^{ib} \\ \dot{\xi}_{12}^{ib} = [-6, 0.1, 0.3]\chi_i^b + [1, -1]\tilde{u}_i \\ \dot{\xi}_{21}^{ib} = 0.2\chi_i^a + [1, -0.1, 0.9]\chi_i^b + [1, 1]\tilde{u}_i \\ y_i = \text{col}(\xi_{11}^{ib}, \xi_{21}^{ib}) \end{cases}$$

where $\tilde{R} = [1, -1; 1, 1]$.

Thus, the whole controller is given as follows:

$$\begin{cases} \tilde{u}_i = \begin{bmatrix} -0.1\hat{\chi}_i^a + 2.5\hat{\xi}_{11}^{ib} - 0.6\hat{\xi}_{21}^{ib} + 0.5u_{i1} + 0.5u_{i2} \\ -0.1\hat{\chi}_i^a - 3.5\hat{\xi}_{11}^{ib} - 0.3\hat{\xi}_{21}^{ib} + 0.1\hat{\xi}_{12}^{ib} - 0.5u_{i1} + 0.5u_{i2} \end{bmatrix} \\ u_{i1} = v_{i1}, \quad u_{i2} = v_{i2} \\ v_i = \begin{bmatrix} v_{i1} \\ v_{i2} \end{bmatrix} = -\frac{1}{\varepsilon^2} [c_0(y_i - z_i) + \varepsilon c_1 y_i^{(1)}] \\ \dot{z}_i = -\nabla f_i(y_i) - \sum_{j=1}^N a_{ij}(\lambda_i - \lambda_j) \\ \dot{\lambda}_i = \sum_{j=1}^N a_{ij}(z_i - z_j) \end{cases}$$

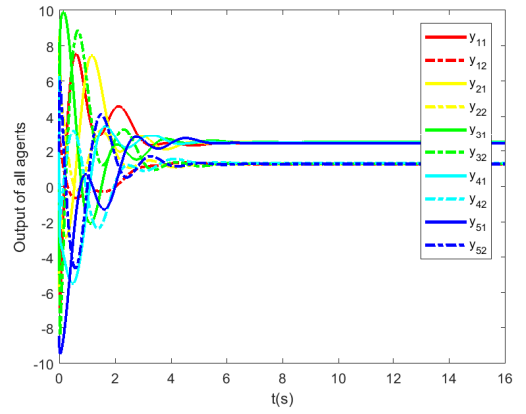


Fig. 5. Profiles of all components in y_i under output feedback control (31).

where $\hat{y}_i = \text{col}(\hat{\xi}_{11}^{ib}, \hat{\xi}_{21}^{ib})$, $\hat{y}_i^{(1)} = \text{col}(\hat{\xi}_{12}^{ib}, \hat{u}_{i2})$, and $\hat{\chi}_i^a$ are generated by the following observer:

$$\begin{cases} \dot{\hat{y}}_i = \hat{y}_i^{(1)} - l_1(\hat{y}_i - y_i) \\ \dot{\hat{y}}_i^{(1)} = v_i - l_2(\hat{y}_i - y_i) \\ \dot{\hat{\chi}}_i^a = -3\hat{\chi}_i^a + [-3.5, 0.3, -2]\hat{\chi}_i^b. \end{cases}$$

Take $c_0 = 4$, $c_1 = 8$, $l_1 = 4$, $l_2 = 8$, $\varepsilon = 1$, and all initials are generated (randomly) in $[-10, 10]^{14}$. This output optimal consensus problem can be solved by the output feedback controller (31). The simulation results are given in Figs. 4 and 5 and all the outputs converge to the global optimal point.

VI. CONCLUSION

This paper has investigated the optimal output consensus problem for high-order minimum-phase multiagent systems. An embedded control scheme has been proposed and applied to solve this problem based on the introduction of an optimal signal generator. The proposed algorithms have been proved to converge to the optimal solution asymptotically or exponentially with different conditions. In fact, many challenging optimal output consensus problems remain to be done, including the cases of practical and nonlinear agents, or various uncertainties from communication or environment, or different optimization constraints.

REFERENCES

- [1] J. A. Fax and R. M. Murray, "Information flow and cooperative control of vehicle formations," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1465–1476, Sep. 2004.
- [2] X. Wang, W. Ni, and X. Wang, "Leader-following formation of switching multirobot systems via internal model," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 42, no. 3, pp. 817–826, Jun. 2012.
- [3] Y. Su and J. Huang, "Cooperative output regulation with application to multi-agent consensus under switching network," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 42, no. 3, pp. 864–875, Jun. 2012.
- [4] Y. Tang, Y. Hong, and X. Wang, "Distributed output regulation for a class of nonlinear multi-agent systems with unknown-input leaders," *Automatica*, vol. 62, pp. 154–160, Dec. 2015.
- [5] H. Rezaee and F. Abdollahi, "Average consensus over high-order multiagent systems," *IEEE Trans. Autom. Control*, vol. 60, no. 11, pp. 3047–3052, Nov. 2015.
- [6] J. H. Seo, H. Shim, and J. Back, "Consensus of high-order linear systems using dynamic output feedback compensator: Low gain approach," *Automatica*, vol. 45, no. 11, pp. 2659–2664, 2009.

- [7] J. Xi, Z. Shi, and Y. Zhong, "Output consensus analysis and design for high-order linear swarm systems: Partial stability method," *Automatica*, vol. 48, no. 9, pp. 2335–2343, 2012.
- [8] W. Ren and E. Atkins, "Distributed multi-vehicle coordinated control via local information exchange," *Int. J. Robust Nonlin. Control*, vol. 17, nos. 10–11, pp. 1002–1033, 2007.
- [9] C.-Q. Ma and J.-F. Zhang, "Necessary and sufficient conditions for consensusability of linear multi-agent systems," *IEEE Trans. Autom. Control*, vol. 55, no. 5, pp. 1263–1268, May 2010.
- [10] A. Nedic, A. Ozdaglar, and P. A. Parrilo, "Constrained consensus and optimization in multi-agent networks," *IEEE Trans. Autom. Control*, vol. 55, no. 4, pp. 922–938, Apr. 2010.
- [11] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein, "Distributed optimization and statistical learning via the alternating direction method of multipliers," *Found. Trends Mach. Learn.*, vol. 3, no. 1, pp. 1–122, 2011.
- [12] S. Bose, S. H. Low, T. Teeraratkul, and B. Hassibi, "Equivalent relaxations of optimal power flow," *IEEE Trans. Autom. Control*, vol. 60, no. 3, pp. 729–742, Mar. 2015.
- [13] Y. Zhang, Y. Lou, Y. Hong, and L. Xie, "Distributed projection-based algorithms for source localization in wireless sensor networks," *IEEE Trans. Wireless Commun.*, vol. 14, no. 6, pp. 3131–3142, Jun. 2015.
- [14] D. Yuan, S. Xu, and H. Zhao, "Distributed primal–dual subgradient method for multiagent optimization via consensus algorithms," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 41, no. 6, pp. 1715–1724, Dec. 2011.
- [15] P. Yi and Y. Hong, "Quantized subgradient algorithm and data-rate analysis for distributed optimization," *IEEE Trans. Control Netw. Syst.*, vol. 1, no. 4, pp. 380–392, Dec. 2014.
- [16] P. Lin, W. Ren, and Y. Song, "Distributed multi-agent optimization subject to nonidentical constraints and communication delays," *Automatica*, vol. 65, pp. 120–131, Mar. 2016.
- [17] G. Shi, K. H. Johansson, and Y. Hong, "Reaching an optimal consensus: Dynamical systems that compute intersections of convex sets," *IEEE Trans. Autom. Control*, vol. 58, no. 3, pp. 610–622, Mar. 2013.
- [18] S. S. Kia, J. Cortés, and S. Martínez, "Distributed convex optimization via continuous-time coordination algorithms with discrete-time communication," *Automatica*, vol. 55, pp. 254–264, May 2015.
- [19] X. Wang, Y. Hong, and H. Ji, "Distributed optimization for a class of nonlinear multiagent systems with disturbance rejection," *IEEE Trans. Cybern.*, vol. 46, no. 7, pp. 1655–1666, Jul. 2016.
- [20] Q. Liu and J. Wang, "A projection neural network for constrained quadratic minimax optimization," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 26, no. 11, pp. 2891–2900, Nov. 2015.
- [21] P. Yi, Y. Hong, and F. Liu, "Distributed gradient algorithm for constrained optimization with application to load sharing in power systems," *Syst. Control Lett.*, vol. 83, pp. 45–52, Sep. 2015.
- [22] Q. Liu and J. Wang, "A second-order multi-agent network for bound-constrained distributed optimization," *IEEE Trans. Autom. Control*, vol. 60, no. 12, pp. 3310–3315, Dec. 2015.
- [23] Z. Qiu, S. Liu, and L. Xie, "Distributed constrained optimal consensus of multi-agent systems," *Automatica*, vol. 68, pp. 209–215, Jun. 2016.
- [24] S. Yang, Q. Liu, and J. Wang, "A multi-agent system with a proportional-integral protocol for distributed constrained optimization," *IEEE Trans. Autom. Control*, vol. 62, no. 7, pp. 3461–3467, Jul. 2017.
- [25] C.-Y. Kim, D. Song, Y. Xu, J. Yi, and X. Wu, "Cooperative search of multiple unknown transient radio sources using multiple paired mobile robots," *IEEE Trans. Robot.*, vol. 30, no. 5, pp. 1161–1173, Oct. 2014.
- [26] E. Dall'Anese, H. Zhu, and G. B. Giannakis, "Distributed optimal power flow for smart microgrids," *IEEE Trans. Smart Grid*, vol. 4, no. 3, pp. 1464–1475, Sep. 2013.
- [27] Y. Zhang and Y. Hong, "Distributed optimization design for second-order multi-agent systems," in *Proc. IEEE 33rd Chin. Control Conf. (CCC)*, Nanjing, China, 2014, pp. 1755–1760.
- [28] Z. Deng and Y. Hong, "Multi-agent optimization design for autonomous Lagrangian systems," *Unmanned Syst.*, vol. 4, no. 1, pp. 5–13, 2016.
- [29] Y. Zhang and Y. Hong, "Distributed optimization design for high-order multi-agent systems," in *Proc. IEEE 34th Chin. Control Conf. (CCC)*, Hangzhou, China, 2015, pp. 7251–7256.
- [30] R. T. Rockafellar, *Convex Analysis*. Princeton, NJ, USA: Princeton Univ., 1970.
- [31] D. P. Bertsekas, A. Nedic, and A. E. Ozdaglar, *Convex Analysis and Optimization*. Belmont, MA, USA: Athena Sci., 2003.
- [32] C. Godsil and G. F. Royle, *Algebraic Graph Theory*. New York, NY, USA: Springer, 2001.
- [33] P. Falb and W. Wolovich, "Decoupling in the design and synthesis of multivariable control systems," *IEEE Trans. Autom. Control*, vol. AC-12, no. 6, pp. 651–659, Dec. 1967.
- [34] A. Isidori, *Nonlinear Control Systems*, (Communications and Control Engineering Series), 3rd ed. London, U.K.: Springer, 1995.
- [35] S. Bhattacharya and V. Kumar, "Distributed optimization with pairwise constraints and its application to multi-robot path planning," in *Proc. Robot. Sci. Syst. VI*, 2011, p. 177.
- [36] J. C. Derenick and J. R. Spletzer, "Convex optimization strategies for coordinating large-scale robot formations," *IEEE Trans. Robot.*, vol. 23, no. 6, pp. 1252–1259, Dec. 2007.
- [37] Y. Nesterov, *Introductory Lectures on Convex Optimization: A Basic Course*, vol. 87. New York, NY, USA: Springer, 2013.
- [38] D. Hristu-Varsakelis and W. S. Levine, *Handbook of Networked and Embedded Control Systems*. Boston, MA, USA: Birkhäuser, 2005.
- [39] C.-T. Chen, *Linear System Theory and Design*. New York, NY, USA: Oxford Univ. Press, 1995.
- [40] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Upper Saddle River, NJ, USA: Prentice-Hall, 2002.
- [41] P. A. Ioannou and J. Sun, *Robust Adaptive Control*. Upper Saddle River, NJ, USA: Prentice-Hall, 1995.
- [42] R. Alur *et al.*, "Hierarchical modeling and analysis of embedded systems," *Proc. IEEE*, vol. 91, no. 1, pp. 11–28, Jan. 2003.



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