

Distributed coordination for a class of non-linear multi-agent systems with regulation constraints

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Abstract: In this study, a multi-agent coordination problem with steady-state regulation constraints is investigated for a class of non-linear systems. Unlike the existing leader-following coordination formulations, a reference signal is not given by a dynamic autonomous leader but determined as the optimal solution of a distributed optimisation problem. Furthermore, the authors consider a global constraint having noisy data observations for the optimisation problem, which implies that the reference signal is not trivially available with the existing optimisation algorithms. To handle these challenges, the authors present a passivity-based analysis and design approach by using only local objective function, local data observation and exchanged information from their neighbours. The proposed distributed algorithms are shown to achieve the optimal steady-state regulation by rejecting the unknown observation disturbances for passive non-linear agents, which are persuasive in various practical problems. Applications and simulation examples are then given to verify the effectiveness of the proposed design.

1 Introduction

Over the last decade, there has been an increasing percentage of literature concerning the coordination problem of multi-agent systems due to its wide applications in engineering systems. e.g. multi-robot system and wireless network (see [1–3] and references therein). As one fundamental problem of this topic, leader-following coordination has been widely studied [4–8]. In this problem, a (virtual) leader is often set up to generate reference signals for each agent to follow, while this leader is usually given as a known dynamic system with possible unknown states. Then the main task is to determine the agents' controllers, which should only utilise local information, such that the resultant state or output trajectories of the agents can track the reference signal generated by the leader. This leader-following formulation has been extensively embodied into various effective algorithms for practical engineering problems, e.g. formation control and attitude synchronisation [2, 9].

Here, we follow this line but consider a particular case when the reference signal is not generated as the trajectory of an autonomous leader, but as the unknown optimal solution of distributed optimisation problems. This type of problem arises naturally from many practical applications. For example, in a source seeking problem, we aim to control one or more agents with non-linear dynamics to seek the extremum of some unknown signal fields based on local signal measurements. Thus the reference (although a constant) is neither available in advance nor can be generated by an autonomous leader without real-time measurements and computations. Many other practical engineering applications have a similar feature that the reference signal is a (time-varying) maximum or minimum of some performance functions, e.g. the design of anti-lock braking systems [10], optimal rendezvous of unmanned aerial vehicles [11]. Moreover, practically each agent can only access noisy local observations/measurements. How to solve these optimal regulation problems over multi-agent systems with data locality and impurity is important and challenging.

Furthermore, the so-called distributed optimisation becomes more and more popular with its broad applications in various fields such as intersection computation [12] and smart grids [13]. In this problem, each agent only knows its own local cost function, and may only know its locally observable data. The agents aim to cooperatively achieve a consensus on their states and optimise the sum of all local cost functions. Both discrete-time and continuous-

time gradient-based optimisation algorithms were proposed [14–17]. In fact, we can regard this problem as a leader-following problem when the agents are single integrators and the reference signal is determined as the optimal solution. We may wonder its solvability when agents are of high-order physical processes, e.g. motion of mobile sensors and the dynamics of the inventory system. Since the decision variables are determined as outputs of high-order non-linear dynamic systems that essentially cannot be treated as single integrators, the solvability of those distributed optimisation problems over physical dynamics can be much more challenging than the conventional cases.

Based on these observations, we aim to formulate and investigate a distributed coordination problem over high-order non-linear multi-agent systems where the reference signal is determined as the optimal solution to a constrained optimisation problem. Specifically, we consider a class of passive non-linear agents, and the output variables are regulated according to a distributed resource allocation problem. The goal is then to regulate these outputs to the optimal solution of the associated optimisation problem in a distributed way.

In fact, very few optimisation results have been obtained on this topic with agents with high-order dynamics, though there are plenty of conclusions obtained for integrator-type agents as mentioned above. Since practical systems are hardly described by single integrators, we have to take high-order dynamics into consideration. For example, for double integrators, Zhang and Hong [18] proposed a distributed optimisation algorithm with an integral control idea and a similar design was employed for Lagrangian agents [19]. Distributed optimisation with input disturbances was also considered in [20] by an internal-model approach for a class of non-linear minimum phase agents. An important engineering problem related to this topic is the economic dispatch in power systems. While economic dispatch can be formulated and solved as a distributed optimisation problem [21, 22], frequency dynamics were taken into consideration in [23] and an optimisation requirement on the steady-state inputs should be satisfied. An internal-model based controller was proposed to solve the optimal frequency regulation problem in power grids under unknown and possible time-varying load changes. Generally, resource allocation over non-linear multi-agent systems is still far from being solved.

In view of the aforementioned results, the main contributions of this study are at least two-fold:

(i) A general multi-agent coordination problem with regulation constraints is formulated for a class of non-linear agents. Since the steady-state of each agent cannot be determined in a centralised way and can only be determined asymptotically through a distributed computation within the networks, it can be taken as distributed extensions of conventional steady-state regulation problems [10, 24]. Distributed algorithms are proposed to solve this problem by using only local data and exchanged information from their neighbours, along with both the asymptotic and exponential convergence results, while only local and asymptotic results were considered in [24].

(ii) A passivity-based approach is adopted to distributedly solve the resource allocation problem over high-order non-linear multi-agent systems. When agents are single integrators without observation disturbances, this problem reduces to the conventional distributed resource allocation problem [21, 22]. Note that the decision variables are outputs of physical plants with high-order non-linear dynamics, and hence this problem is much more challenging than traditional cases. This work shows that the passivity-based approach could provide a fresh design methodology to solve this kind of problem for high-order dynamic agents, including distributed inventory control [25] and average consensus problem [2, 26].

Additionally, unknown data observation disturbances are considered in our formulation and then rejected by an observer-based compensator. This approach is different from the existing internal model-based designs [20, 23, 27], and may provide new perspectives and approaches to deal with this kind of problem.

The paper is organised as follows. Preliminaries about convex analysis, graph theory and passivity are given in Section 2. Then the problem of distributed coordination with regulation constraints is formulated for a class of non-linear multi-agent systems in Section 3. Main results are presented and proved in Section 4 along with the proposed gradient-based controls. Following that, three applications are given with discussions in Section 5 to illustrate the applicability and effectiveness of the proposed algorithm. Finally, concluding remarks are given in Section 6.

Notations: let \mathbb{R}^n be the n -dimensional Euclidean space. For a vector x , $\|x\|$ denotes its Euclidean norm. $\mathbf{1}_N$ (and $\mathbf{0}_N$) denotes an N -dimensional all-one (and all-zero) column vector. $\text{col}(a_1, \dots, a_n) = [a_1^\top, \dots, a_n^\top]^\top$ for column vectors a_i ($i = 1, \dots, n$). $r_N = (1/\sqrt{N})\mathbf{1}_N$, and $R_N \in \mathbb{R}^{N \times (N-1)}$ satisfying $R_N^\top r_N = \mathbf{0}_N$, $R_N^\top R_N = I_{N-1}$ and $R_N R_N^\top = I_N - r_N r_N^\top$.

2 Preliminaries

In this section, preliminaries are given about convex analysis [28], graph theory [29] and system passivity [30].

2.1 Convex analysis

A function $f(\cdot): \mathbb{R}^N \rightarrow \mathbb{R}$ is said to be convex if for any $0 \leq a \leq 1$,

$$f(a\zeta_1 + (1-a)\zeta_2) \leq af(\zeta_1) + (1-a)f(\zeta_2), \quad \forall \zeta_1, \zeta_2 \in \mathbb{R}^N.$$

A differentiable function f is convex over \mathbb{R}^N if

$$f(\zeta_1) - f(\zeta_2) \geq \nabla f(\zeta_2)^\top (\zeta_1 - \zeta_2), \quad \forall \zeta_1, \zeta_2 \in \mathbb{R}^N \quad (1)$$

and f is strictly convex over \mathbb{R}^N if the above inequality is strict whenever $\zeta_1 \neq \zeta_2$, and f is ω -strongly convex ($\omega > 0$) over \mathbb{R}^N if $\forall \zeta_1, \zeta_2 \in \mathbb{R}^N$

$$(\nabla f(\zeta_1) - \nabla f(\zeta_2))^\top (\zeta_1 - \zeta_2) \geq \omega \|\zeta_1 - \zeta_2\|^2. \quad (2)$$

A function $f: \mathbb{R}^N \rightarrow \mathbb{R}^N$ is Lipschitz with constant $M > 0$, or simply M -Lipschitz, if

$$\|f(\zeta_1) - f(\zeta_2)\| \leq M \|\zeta_1 - \zeta_2\|, \quad \forall \zeta_1, \zeta_2 \in \mathbb{R}^N.$$

2.2 Graph theory

A weighted undirected graph is described by $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$ with a node set $\mathcal{N} = \{1, \dots, N\}$ and an edge set \mathcal{E} . $(i, j) \in \mathcal{E}$ denotes an edge between nodes i and j . The weighted adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ is defined by $a_{ii} = 0$ and $a_{ij} = a_{ji} \geq 0$ ($a_{ij} > 0$ if and only if there is an edge between nodes i and j). The neighbour set of node i is defined as $\mathcal{N}_i = \{j: (j, i) \in \mathcal{E}\}$ for $i = 1, \dots, n$. A path in graph \mathcal{G} is an alternating sequence $i_1 e_1 i_2 e_2 \dots e_{k-1} i_k$ of nodes i_l and edges $e_m = (i_m, i_{m+1}) \in \mathcal{E}$ for $l = 1, 2, \dots, k$. If there is a path between any two vertices of a graph \mathcal{G} , then the graph is said to be connected. The Laplacian $L = [l_{ij}] \in \mathbb{R}^{N \times N}$ of graph \mathcal{G} is defined as $l_{ii} = \sum_{j \neq i} a_{ij}$ and $l_{ij} = -a_{ij}$ ($j \neq i$), which is thus symmetric. Denote the eigenvalues of Laplacian matrix L associated with an undirected graph \mathcal{G} as $\lambda_1 \leq \dots \leq \lambda_N$.

2.3 System passivity

Passivity, due to its explicit physical meaning and simplicity to manipulate, has been extensively discussed in controlling various practical engineering systems (see [30] and references therein). Usually, only the case when the equilibrium point is zero is investigated. However, when desirable regulation points are specified as solutions to optimisation problems, we do not know the optimal point beforehand and have to go back to the general case with non-zero equilibrium points, which has been named as incremental passivity in the literature [31–34].

Consider a dynamic system of the following form:

$$\dot{x} = g(x, u), \quad y = h(x), \quad x \in \mathbb{R}^n, u, y \in \mathbb{R}^p. \quad (3)$$

Let $\mathbb{E} = \{(x^*, u^*) \mid g(x^*, u^*) = 0\}$ be the equilibrium points of this system. System (3) is said to be passive with respect to (w.r.t.) (x^*, u^*) if there exist two functions $\alpha_1(\cdot), \alpha_2(\cdot) \in \mathcal{K}_\infty$ and a continuously differentiable storage function $V(x, x^*)$ satisfying that

$$(i) \alpha_1(\|x - x^*\|) \leq V(x, x^*) \leq \alpha_2(\|x - x^*\|).$$

$$(ii) \dot{V} \leq (y - y^*)^\top (u - u^*) \text{ with } y^* = h(x^*),$$

where \dot{V} is short for $\dot{V}(x, x^*) \triangleq (\partial V / \partial x)(x, x^*) \dot{x}$.

When the second condition is strengthened as

$$\dot{V} \leq -\alpha_3(\|x - x^*\|) + (y - y^*)^\top (u - u^*)$$

for some function $\alpha_3(\cdot) \in \mathcal{K}$, this system is said to be strictly passive w.r.t. (x^*, u^*) . If $\alpha_1(\cdot), \alpha_2(\cdot), \alpha_3(\cdot)$ can be taken as quadratic functions, this system is said to be exponentially passive w.r.t. (x^*, u^*) . For consistence, a memoryless function $\phi: \mathbb{D} \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ is passive w.r.t. $y^* \in \mathbb{R}^n$ if it satisfies

$$(y - y^*)^\top (\phi(y) - \phi(y^*)) \geq 0, \quad \forall y \in \mathbb{D}. \quad (4)$$

When the equality occurs only if $y = y^*$, we say it is strictly passive w.r.t. y^* . Additionally, when there exists a constant $\gamma > 0$ such that

$$(y - y^*)^\top (\phi(y) - \phi(y^*)) \geq \gamma \|y - y^*\|^2, \quad \forall y \in \mathbb{D},$$

it is exponentially passive with modulus γ w.r.t. y^* .

In this study, we only consider the single-input single-output case, i.e. $p = 1$, while the following arguments hold for the multi-input multi-output case with $p > 1$ as well. For a given passive system w.r.t. (x^*, u^*) , this equilibrium is said to be assignable if there exists a passive $\phi(\cdot)$ satisfying $\phi(y^*) + u^* = 0$. The following

lemma presents an important approach to stabilise non-linear passive systems.

Lemma 1: Suppose system (3) is passive w.r.t. an assignable equilibrium (x^*, u^*) , then, $x = x^*$ is Lyapunov stable under $u = -\phi(y)$. If $\phi(\cdot)$ is strictly passive w.r.t. y^* , we have $y(t) \rightarrow y^*$ as t goes to infinity. Moreover, the trajectory x converges to x^* exponentially fast if system (3) is exponentially passive w.r.t. (x^*, u^*) .

Proof: The proof follows standard Lyapunov arguments. In fact, when the system is passive w.r.t. (x^*, u^*) , we have a continuously differentiable storage function $V(x, x^*)$ and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ satisfying: $\alpha_1(\|x - x^*\|) \leq V(x, x^*) \leq \alpha_2(\|x - x^*\|)$ and $\dot{V} \leq (y - y^*)^\top (u - u^*)$ with $y^* = h(x^*)$.

By taking $u = -\phi(y)$, we have $\dot{V} \leq -(y - y^*)^\top (\phi(y) - \phi(y^*))$. From the strict passivity of $\phi(\cdot)$, it implies $\dot{V} \leq 0$ and the equation happens only if $y = y^*$. By LaSalle's invariance principle [30] and the smoothness of $h(\cdot)$, we can obtain $x(t)$ converges to the largest invariant set contained in $\{x \in \mathbb{R}^n | h(x) = h(x^*)\}$ as $t \rightarrow +\infty$, which implies $y(t) \rightarrow y^*$ as t goes to infinity.

When this system is exponentially passive, we further have $k_1 \|x - x^*\|^2 \leq V(x, x^*) \leq k_2 \|x - x^*\|^2$ and $\dot{V} \leq -k_3 \|x - x^*\|^2$, which imply the exponential convergence of x w.r.t. x^* . \square

By this lemma, the asymptotic regulation of y to y^* is transformed into a problem that is to find a passive function $\phi(y)$ satisfying $\phi(y^*) + u^* = 0$. For a special case when $x^* = 0, u^* = 0$, we need to find a monotone (passive) function $\phi(\cdot)$ vanishing at the origin, which is consistent with the existing results. It is well known that every continuous (strictly, strongly) convex function has its derivative (including gradients as its special cases) as an associated (strictly, exponentially) passive function w.r.t. the minimum point [35]. This observation will play a key motivation for our gradient-based algorithm design when we do not have the direct information of x^* and hence y^* . Although we consider gradient-based designs, various functions (not limited to gradients) can be employed in a passivity-based design for different problems, e.g. skew-symmetric linear operators and saddle-point operators [35, 36].

3 Problem formulation

Consider N agents with dynamics of the form

$$\dot{x}_i = g_i(x_i, u_i), \quad y_i = h_i(x_i), \quad i = 1, \dots, N \quad (5)$$

with state variable $x_i \in \mathbb{R}^{n_i}$, control input $u_i \in \mathbb{R}$, and output $y_i \in \mathbb{R}$. The functions $g_i(\cdot)$ and $h_i(\cdot)$ are assumed to be smooth.

Along with node dynamics, the i th agent has a local cost function $f_i(y_i)$. For this multi-agent system, we associate it an optimisation problem with coupled constraints as follows:

$$\begin{aligned} \text{minimize} \quad & f(y) \triangleq \sum_{i=1}^N f_i(y_i), \\ \text{subject to} \quad & \sum_{i=1}^N y_i = \sum_{i=1}^N d_i^0, \end{aligned} \quad (6)$$

where d_i^0 can only be obtained by agent i by local measurements. This optimisation problem is often called resource allocation [37] and many practical applications can be formulated as the above, e.g. economic dispatch in power systems [22], flow control in networks [37]. We can regard y_i as the amount of resource located at node i and interpret $-f_i(y_i)$ as the local (concave) utility function. Coupled with the physical agents, we aim to design proper controllers such that the outputs of these agents asymptotically solve the optimisation problem (6). This implies the above optimisation problem is a requirement on the steady-state of this multi-agent system. In other words, the controller should regulate the systems' outputs such that the equality constraint is

satisfied and optimal performance is achieved, both in an asymptotic manner.

Moreover, we are interested in distributed algorithms without setting up a centralised working station which might be expensive and inhibitive in some circumstances. Namely, we aim to find a distributed protocol using only local objective function, constraint-related observation, and exchanged information from their neighbours to drive the outputs of agents to reach an allocation that maximises the total utility $-\sum_{i=1}^N f_i(y_i)$. For this purpose, an undirected graph \mathcal{G} is employed to describe the information sharing relationships among those nodes represented by $\mathcal{N} = \{1, \dots, N\}$. If nodes i and j can exchange information with each other, then there is an edge (i, j) in the graph \mathcal{G} , i.e. $a_{ij} = a_{ji} > 0$.

Furthermore, we assume agent i can only get a polluted observation $d_i(t) \triangleq d_i^0 + d_i^e(t)$ of d_i^0 by an imperfect sensor, where the disturbance is assumed consisting of k_i sinusoidal signals with distinct but known frequencies: $\omega_{i1}, \dots, \omega_{ik_i}$. In fact, this type of disturbance can produce a fair approximation of any bounded periodic disturbance signal by summing up the dominated harmonics in its Fourier series expansion and has been used as typical non-trivial disturbances in the control literature [38, 39].

We then formulate the distributed coordination problem for non-linear multi-agent systems with steady-state regulation constraints as follows. Given the graph \mathcal{G} , cost function $f_i(\cdot)$ and dynamic plant (5), find a distributed control u_i for agent i , which only depends on its own local data and exchanged information from its neighbours, such that the trajectories of agents are bounded and satisfy

$$\lim_{t \rightarrow +\infty} y_i = y_i^*, \quad \text{for } i = 1, \dots, N$$

where $\text{col}(y_1^*, \dots, y_N^*)$ is the optimal solution of (6).

Remark 1: Unlike the existing leader-following coordination problems in multi-agent systems [2, 4, 7], the reference signal cannot be modelled as an autonomous leader. In fact, the desired steady states can only be obtained by a distributed cooperation and computation. On the other hand, when $f_i(y_i) = (1/2)y_i^2$ and $d_i^0 = y_i(0)$, the optimal solution of (6) is $(\sum_{i=1}^N y_i(0)/N)\mathbf{1}_N$. Our formulation provides another way to solve the well-known average consensus problem for non-linear dynamic agents [2, 26, 40].

Our formulation can be taken as a constrained optimisation problem subject to noisy constraint data observations. Compared with the traditional stochastic or the worst-case formulation in robust optimisation, the observation perturbations/uncertainties here are modelled as structured but unknown ones, which can be deemed as a balance on available information of perturbations between its nominal version and stochastic/worst-case version.

To achieve a coordination among these agents, some technical assumptions are needed.

Assumption 1: For $i = 1, \dots, N$, the function $f_i: \mathbb{R} \rightarrow \mathbb{R}$ is convex, twice continuously differentiable with bounded Hessian, i.e. there exist $0 < \underline{h}_i \leq \bar{h}_i < \infty$ such that, for all i :

$$\underline{h}_i \leq \nabla^2 f_i(s) \leq \bar{h}_i, \quad \forall s \in \mathbb{R}.$$

Assumption 2: The communication graph \mathcal{G} is connected.

The assumptions have been widely used in many publications [8, 29, 41]. Assumption 1 is made to guarantee the solvability of this optimisation problem. In fact, it implies the Lipschitz continuity of ∇f_i and strong convexity of f_i . Then, the optimisation problem (6) is solvable and has a unique solution $y^* = \text{col}(y_1^*, \dots, y_N^*)$ by Proposition 3.2.1 in [28]. It is well known [29] that under Assumption 2, the associated Laplacian L of this graph is symmetric with rank $N - 1$, and its null space is spanned by $\mathbf{1}_N$.

Another technical assumption is made as follows.

Assumption 3: For any $i \in \{1, \dots, N\}$ and constant output r , there exist two unique smooth functions $\mathbf{x}_i(\cdot)$ and $\mathbf{u}_i(\cdot)$ satisfying

$$g_i(\mathbf{x}_i(r), \mathbf{u}_i(r)) = 0, \quad r = h_i(\mathbf{x}_i(r)). \quad (7)$$

Furthermore, the function $\mathbf{u}_i(\cdot)$ is M_i -Lipschitz at its arguments on the concerned set.

Note that our problem is essentially an asymptotic regulation problem where the reference is determined by the optimisation problem (6), thereby, Assumption 3 can be understood as the existence of the solution to regulation equations in the terminology of output regulation. The Lipschitzness of $\mathbf{u}_i(\cdot)$ is only made for technical analysis, and it naturally holds when the concerned sets are compact. Similar assumptions can be found in [30, 38].

Denoting $x_i^* = \mathbf{x}_i(y_i^*)$ and $u_i^* = \mathbf{u}_i(y_i^*)$, we then focus on a class of passive non-linear dynamic systems as follows.

Assumption 4: For $i = 1, \dots, N$, the dynamics (5) is passive w.r.t. (y_i^*, x_i^*, u_i^*) , i.e. there exists a continuously differentiable storage function $V_i(x_i, x_i^*)$ satisfying $\alpha_{i1}(\|x_i - x_i^*\|) \leq V_i(x_i, x_i^*) \leq \alpha_{i2}(\|x_i - x_i^*\|)$ for two \mathcal{K}_∞ functions $\alpha_{i1}(\cdot)$ and $\alpha_{i2}(\cdot)$ such that

$$\dot{V}_i \leq (y_i - y_i^*)^\top (u_i - u_i^*).$$

As mentioned before, this condition is an extended version of a classical passivity property w.r.t. non-zero equilibrium (by non-zero inputs). In fact, it has been termed as *incremental passivity* property in many publications [31–34], and a large class of typical systems falls into this class perhaps after an inner feedback passivation loop. Notice that Assumption 4 only concerns with the dynamics of the non-linear agents and imposes no restrictions on the optimisation problem.

In the following section, we solve the distributed coordination problem with regulation constraints by passivity-based arguments.

4 Main results

In this section, we propose a distributed algorithm to solve the distributed coordination problem determined by (5) and (6), and then prove its stability via passivity techniques.

First, we split the control efforts into two parts $u_i = u_i^1 + u_i^2$, where u_i^1 is designed for distributed optimisation with disturbance rejection, and u_i^2 for asymptotic steady-state regulation.

Inspired by works [22, 37, 41], the optimisation problem (6) can be rewritten as a monotropic programming problem and solved by a primal-dual approach if d^0 is known. Since there are observation disturbances for agent i in d_i^0 , an observer-based approach is employed to estimate d_i^0 while rejecting those disturbances d_i^e . Then, the first part of our control is given as follows:

$$\begin{aligned} u_i^1 &= -\gamma \nabla f_i(y_i) + \lambda_i, \\ \dot{\lambda}_i &= -\lambda_i^y - z_i^y + d_i - y_i - D_i^e \eta_i, \\ \dot{\eta}_i &= (S_i - L_i D_i) \eta_i + L_i d_i, \\ \dot{z}_i &= \lambda_i^y, \quad i = 1, \dots, N, \end{aligned} \quad (8)$$

where

$$S_i = \text{diag} \left(0, \begin{bmatrix} 0 & \omega_{i1} \\ -\omega_{i1} & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & \omega_{ik_i} \\ -\omega_{ik_i} & 0 \end{bmatrix} \right),$$

$$D_i = [1, \underbrace{1, 0, \dots, 1, 0}_{2k_i}],$$

$$D_i^e = [\underbrace{1, 0, \dots, 1, 0}_{2k_i}], \quad \lambda_i^y \triangleq \sum_{j=1}^N a_{ij}(\lambda_i - \lambda_j), \quad z_i^y \triangleq \sum_{j=1}^N a_{ij}(z_i - z_j),$$

and $\gamma \in \mathbb{R}$, $L_i \in \mathbb{R}^{(2k_i+1) \times 1}$ will be determined later. Here, the parameter γ can be regarded as a proportional gain to correct the

regulation error, and the η_i -subsystem is a local observer to estimate $d_i(t)$ for disturbance rejection.

The following lemma guarantees the selection of gain matrix L_i such that $S_i - L_i D_i$ is Hurwitz and the effectiveness of our algorithm in disturbance rejection.

Lemma 2: The pair (S_i, D_i) is observable.

Proof: We consider the rank of $[S_i^\top - \lambda I_{2k_i+1}, D_i^\top]$. When $\lambda \notin \{0, \omega_1, \dots, \omega_{k_i}\}$, this is obvious.

Suppose $\lambda = 0$, the matrix $[S_i^\top, D_i^\top]$ can be partitioned as follows:

$$\begin{bmatrix} 0 & \mathbf{0}_{2k_i}^\top & 1 \\ \mathbf{0}_{2k_i} & \hat{S}_i^\top & D_i^{e\top} \end{bmatrix}$$

where

$$\hat{S}_i = \text{diag} \left(\begin{bmatrix} 0 & \omega_{i1} \\ -\omega_{i1} & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & \omega_{ik_i} \\ -\omega_{ik_i} & 0 \end{bmatrix} \right).$$

Since the rank of \hat{S}_i is $2k_i$, $\text{rank}[S_i^\top, D_i^\top]$ is then $2k_i + 1$.

Suppose $\lambda = \omega_{ij}$, without loss of generalities $j = k_i$ and partition $[S_i^\top - \omega_{ik_i}, D_i^\top]$ in the following form:

$$\begin{bmatrix} -\lambda & \mathbf{0}_{2k_i-2}^\top & \mathbf{0}_2^\top & 1 \\ \mathbf{0}_{2k_i-2} & \tilde{S}_i^\top - \lambda I_{2k_i-2} & [\mathbf{0}_{2k_i-2} \ \mathbf{0}_{2k_i-2}] & \tilde{D}_i^{e\top} \\ \mathbf{0}_2 & \begin{bmatrix} \mathbf{0}_{2k_i-2}^\top \\ \mathbf{0}_{2k_i-2}^\top \end{bmatrix} & \begin{bmatrix} -\omega_{iki} & -\omega_{iki} \\ \omega_{iki} & -\omega_{iki} \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix}.$$

where

$$\tilde{S}_i = \text{diag} \left(\begin{bmatrix} 0 & \omega_{i1} \\ -\omega_{i1} & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & \omega_{ik_i-1} \\ -\omega_{ik_i-1} & 0 \end{bmatrix} \right) \quad \text{and} \quad \tilde{D}_i^e = [1, \underbrace{0, \dots, 1, 0}_{2k_i-2}].$$

It can be checked that the rank of

$$\begin{bmatrix} -\omega_{iki} & -\omega_{iki} & 1 \\ \omega_{iki} & -\omega_{iki} & 0 \end{bmatrix}$$

is 2 and the rest part has a rank $2k_i - 1$, thus $\text{rank}[S_i^\top - \omega_{ik_i} I_{2k_i+1}, D_i^\top]$ is then $2k_i + 1$.

To sum up, we obtain $\text{rank}[S_i^\top - \lambda I_{2k_i+1}, D_i^\top] = 2k_i + 1$ holds for any λ . By Popov-Belevitch-Hautus-test [39], this implies the conclusion. \square

For the second part of our control, we recall Assumption 3 and let $u_i^2 = \mathbf{u}_i(y_i)$. The whole control u_i to achieve our designed goal is presented as follows:

$$\begin{aligned} u_i &= u_i^1 + u_i^2 = -\gamma \nabla f_i(y_i) + \lambda_i + \mathbf{u}_i(y_i), \\ \dot{\lambda}_i &= -\lambda_i^y - z_i^y + d_i - y_i - D_i^e \eta_i, \\ \dot{\eta}_i &= (S_i - L_i D_i) \eta_i + L_i d_i, \\ \dot{z}_i &= \lambda_i^y, \end{aligned} \quad (9)$$

where the matrices S_i, D_i, D_i^e, L_i are defined as above.

Under the information sharing constraints, the following lemma shows that at the equilibrium point $(\tilde{x}_i, \tilde{\lambda}_i, \tilde{z}_i)$ of the closed-loop system composed of (5) and (9), the associated output $\tilde{y}_i = h_i(\tilde{x}_i)$

actually solves the resource allocation problem (6) with disturbance rejection.

Lemma 3: Under Assumptions 1–3, the equilibrium point of the closed-loop system composed of (5) and (9) satisfies the following conditions for some constant λ_0 :

$$\nabla_{y_i} f_i(\tilde{y}_i) + \lambda_0 = 0, \quad \sum_{i=1}^N \tilde{y}_i = \sum_{i=1}^N d_i^0, \quad i = 1, \dots, N. \quad (10)$$

Proof: The polluted observation d_i can be rewritten into a form of $d_i(t) = A_{i0} + \sum_{j=1}^{k_i} A_{ij} \sin(\omega_{ij} t + \varphi_{ij})$, where $A_{i0} = d_i^0$, A_{ij} and φ_{ij} are unknown. Then, by taking a proper state variable $\xi_i^d \in \mathbb{R}^{2k_i+1}$, we can put it into $\dot{\xi}_i^d = S_i \xi_i^d$ with $d_i(t) = D_i \xi_i^d$ and an initial condition $\xi_i^d(0)$ determined by A_{ij} and φ_{ij} .

Letting $\bar{d}_i = \eta_i - d_i$ and recalling that $\dot{\eta}_i = (S_i - L_i D_i) \eta_i + L_i d_i$, we have

$$\dot{\bar{d}}_i = \dot{\eta}_i - \dot{\xi}_i^d = (S_i - L_i D_i) \bar{d}_i.$$

By the selection of L_i , the matrix $S_i - L_i D_i$ is Hurwitz and the trajectory of $\bar{d}_i(t)$ goes to 0 as time goes to infinity. Then the equilibrium point of the closed-loop system composed of (5) and (9) can be obtained by setting the derivatives of states to zero, i.e. for $i = 1, \dots, N$

$$g(\tilde{x}_i, \tilde{u}_i) = 0, \quad -\tilde{\lambda}_i^v - \tilde{z}_i^v + d_i^0 - \tilde{y}_i = 0, \quad \tilde{\lambda}_i^v = 0.$$

From $\tilde{\lambda}_i^v = 0$ and by Assumption 2, we have $\tilde{\lambda}_1 = \dots = \tilde{\lambda}_N = \tilde{\lambda}$ for some $\tilde{\lambda}$. By summing up the second equation from 1 to N , it follows $\sum_{i=1}^N \tilde{y}_i = \sum_{i=1}^N d_i^0$, where we use $\mathbf{1}^T L = \mathbf{0}$. By Assumption 3, for given \tilde{y}_i and by the uniqueness of $\mathbf{x}_i(\cdot)$, $\mathbf{u}_i(\cdot)$, we have $-\gamma \nabla f_i(\tilde{y}_i) + \tilde{\lambda}_i + \mathbf{u}_i(\tilde{y}_i) = \mathbf{u}_i(\tilde{y}_i)$ and thus $\nabla f_i(\tilde{y}_i) - \frac{\tilde{\lambda}}{\gamma} = 0$. Let $\lambda_0 = \frac{\tilde{\lambda}}{\gamma}$, the conclusion is thus complete. \square

Since Assumption 1 implies that the optimisation problem (6) has a unique solution, and thus $\tilde{y}_i = y_i^*$, $\mathbf{x}_i(y_i^*) = \mathbf{x}_i(\tilde{y}_i)$, $\mathbf{u}_i(y_i^*) = \mathbf{u}_i(\tilde{y}_i)$ by Assumption 3. Let y_i^* , x_i^* , u_i^* without confusions represent \tilde{y}_i , $\mathbf{x}_i(\tilde{y}_i)$ and $\mathbf{u}_i(\tilde{y}_i)$ to save notations.

It is time to present our first main theorem.

Theorem 1: Under Assumptions 1–4, the distributed coordination problem with regulation constraints determined by (5) and (6) can be solved by the algorithm (9) with $\gamma > \max_i ((1 + M_i)/\underline{h}_i)$ and L_i such that $S_i - L_i D_i$ is Hurwitz, i.e. $\lim_{t \rightarrow +\infty} y_i(t) = y_i^*$ for $i = 1, \dots, N$, where $\text{col}(y_1^*, \dots, y_N^*)$ is the optimal solution of (6). Moreover, if agent i is y_i^* -observable, the closed-loop system is asymptotically stable at its equilibrium point.

Proof: By Lemma 3, we only have to show the stability and output convergence of the closed-loop system w.r.t. its equilibrium point.

Recalling the definition of passivity w.r.t. non-zero equilibrium, the convergence part is trivial if a dynamic system is passive w.r.t. its equilibrium point. We next show the passivity of this closed-loop system with output $y = \text{col}(y_1, \dots, y_N)$ and a new control $\hat{u} \triangleq \text{col}(u_1 - \lambda_1, \dots, u_N - \lambda_N)$.

In fact, since $\bar{S}_i = S_i - L_i D_i$ is Hurwitz, there exists a positive definite matrix $P_i \in \mathbb{R}^{(2k_i+1) \times (2k_i+1)}$, such that $\bar{S}_i^T P_i + P_i \bar{S}_i = -I_{2k_i+1}$. We then consider a candidate storage function $V = \sum_{i=1}^N V_i(x_i, x_i^*) + V_{lc} + \tau \sum_{i=1}^N \bar{d}_i^T P_i \bar{d}_i$, where $V_{lc} \triangleq (1/2)(\lambda_i - \tilde{\lambda}_i)^T (\lambda_i - \tilde{\lambda}_i) + (1/2)(z_i - \tilde{z}_i)^T (z_i - \tilde{z}_i)$ and the constant $\tau > 0$ will be selected later.

It can be verified that item (i) in (2.3) holds. To confirm item (ii), we take the derivative of V along the trajectory of (5) and (9):

$$\begin{aligned} \dot{V} &\leq \sum_{i=1}^N (y_i - y_i^*)^T (u_i - u_i^*) + \sum_{i=1}^N (\lambda_i - \tilde{\lambda}_i)^T \dot{\lambda}_i \\ &\quad + \sum_{i=1}^N (z_i - \tilde{z}_i)^T \dot{z}_i + 2\tau \sum_{i=1}^N \bar{d}_i^T P_i \dot{\bar{d}}_i \\ &= \sum_{i=1}^N (y_i - y_i^*)^T (\hat{u}_i + \lambda_i - u_i^*) + \sum_{i=1}^N (\lambda_i - \tilde{\lambda}_i)^T \dot{\lambda}_i \\ &\quad + \sum_{i=1}^N (z_i - \tilde{z}_i)^T \dot{z}_i + 2\tau \sum_{i=1}^N \bar{d}_i^T P_i \dot{\bar{d}}_i \\ &\leq \sum_{i=1}^N (y_i - y_i^*)^T (\hat{u}_i - \hat{u}_i^*) + \sum_{i=1}^N (y_i - y_i^*)^T (\lambda_i - \tilde{\lambda}_i) + \\ &\quad \sum_{i=1}^N (\lambda_i - \tilde{\lambda}_i)^T \dot{\lambda}_i + \sum_{i=1}^N (z_i - \tilde{z}_i)^T \dot{z}_i - \tau \sum_{i=1}^N \bar{d}_i^T \dot{\bar{d}}_i \\ &\leq (y - y^*)^T (\hat{u} - \hat{u}^*) - (\lambda - \tilde{\lambda})^T L (\lambda - \tilde{\lambda}) \\ &\quad - \sum_{i=1}^N (\lambda_i - \tilde{\lambda}_i) D_i^e \bar{d}_i - \tau \sum_{i=1}^N \bar{d}_i^T \dot{\bar{d}}_i, \end{aligned}$$

where $\lambda = \text{col}(\lambda_1, \dots, \lambda_N)$, $\tilde{\lambda} = \text{col}(\tilde{\lambda}_1, \dots, \tilde{\lambda}_N)$ and $\hat{u}_i(y_i) = -\gamma \nabla f_i(y_i) + \mathbf{u}_i(y_i) - \mathbf{u}_i(y_i^*)$.

From Assumption 2, $(\lambda - \tilde{\lambda})^T L (\lambda - \tilde{\lambda}) \geq c(\lambda - \tilde{\lambda})^T (\lambda - \tilde{\lambda})$, where c is the minimal positive eigenvalue of L . Using Young's inequality to $(\lambda_i - \tilde{\lambda}_i) D_i^e \bar{d}_i$ gives

$$\begin{aligned} \dot{V} &\leq (y - y^*)^T (\hat{u} - \hat{u}^*) - c(\lambda - \tilde{\lambda})^T (\lambda - \tilde{\lambda}) \\ &\quad + \frac{c}{2} (\lambda - \tilde{\lambda})^T (\lambda - \tilde{\lambda}) + \frac{1}{2c} \sum_{i=1}^N \|D_i^e\|^2 \|\bar{d}_i\|^2 - \tau \sum_{i=1}^N \bar{d}_i^T \dot{\bar{d}}_i \\ &\leq (y - y^*)^T (\hat{u} - \hat{u}^*) - \frac{c}{2} (\lambda - \tilde{\lambda})^T (\lambda - \tilde{\lambda}) - (\tau - \tau^*) \|\bar{d}\|^2, \end{aligned}$$

where $\tau^* = (1/2c) \max_i \|D_i^e\|^2$. Letting $\tau \geq 1 + \tau^*$ gives

$$\dot{V} \leq - (y - y^*)^T (\hat{u} - \hat{u}^*) - \frac{c}{2} (\lambda - \tilde{\lambda})^T (\lambda - \tilde{\lambda}) - \|\bar{d}\|^2,$$

which implies the composite system is passive w.r.t. its equilibrium.

Having the passivity of (5) with output y and input \hat{u} , we then prove the convergence of y w.r.t. y^* . For this purpose, we only have to guarantee the strict passivity of \hat{u} w.r.t. y^* by Lemma 1. In fact, from the strong convexity of $f_i(\cdot)$ and the Lipschitzness of $\mathbf{u}_i(\cdot)$ on the concerned set, we have

$$(y_i - y_i^*)^T [\hat{u}_i(y_i) - \hat{u}_i(y_i^*)] \leq (-\gamma \underline{h}_i + M_i) \|y_i - y_i^*\|^2. \quad (11)$$

Taking $\gamma > \max_i ((1 + M_i)/\underline{h}_i)$ gives $(y_i - y_i^*)^T [\hat{u}_i(y_i) - \hat{u}_i(y_i^*)] \leq -\|y_i - y_i^*\|^2$, which implies the strict passivity of $\mathbf{u}_i(\cdot)$ w.r.t. y^* . By Lemma 1, it follows $\lim_{t \rightarrow +\infty} y_i = y_i^*$ for $i = 1, \dots, N$.

To prove the asymptotic stability, we can check that $(x^*, \tilde{\lambda}, z^*, \mathbf{0})$ is the only trajectory contained in the set $\{(x, \lambda, z, \bar{d}) \mid \dot{V} = 0\}$ by the y_i^* -observability of agent i . According to LaSalle's invariance principle [30], one can obtain the conclusions. \square

Notably, the presented passivity-based approach provides a new control perspective for the existing distributed optimisation problems. Unlike the problems considered in [23, 34], we aim to achieve a distributed output optimisation while the optimisation part happens on the input side in their formulations. Since passivity has been widely used in many non-linear control publications [30, 31, 33], this method allows us to consider this problem for more general physical agents other than single integrators [17, 21, 22].

Remark 2: In conventional resource allocation, the plants are actually single integrators (e.g. [21, 22], which are our special

cases of passivity w.r.t. $(y^*, x^*, 0)$. Thus, this conclusion is a non-linear extension of the existing results to a larger class of dynamic systems. Furthermore, as a primal-dual based method to solve the distributed optimisation problem, this algorithm is different from those in [21, 41] which need non-trivial initialisations, and this initialisation-free property makes it more applicable to networked systems with variable numbers of agents.

When the plants are of exponential passivity, one can further obtain the exponential convergence of this algorithm as follows.

Theorem 2: Under the hypothesis of Theorem 1, further assume agent i is exponentially passive w.r.t. (y_i^*, x_i^*, u_i^*) for $1 \leq i \leq N$. Then, the distributed coordination problem with regulation constraints determined by (5) and (6) can be exponentially solved by the algorithm (9) with a properly chosen γ .

Proof: To prove this theorem, we first let $\bar{x}_i = x_i - x_i^* \bar{y}_i = y_i - y_i^*, \bar{\lambda}_i = \lambda_i - \lambda_i^*, \bar{z}_i = z_i - z_i^*$, it follows

$$\begin{aligned}\dot{\bar{\lambda}}_i &= - \sum_{j=1}^N a_{ij}(\bar{\lambda}_i - \bar{\lambda}_j) - \sum_{j=1}^N a_{ij}(\bar{z}_i - \bar{z}_j) + d_i^0 - \bar{y}_i - D_i^e \bar{d}_i, \\ \dot{\bar{d}}_i &= (S_i - L_i D_i) \bar{d}_i, \\ \dot{\bar{z}}_i &= \sum_{j=1}^N a_{ij}(\bar{z}_i - \bar{z}_j).\end{aligned}$$

The whole dynamic can be written in a compact form

$$\begin{aligned}\dot{\bar{\lambda}} &= -L\bar{\lambda} - L\bar{z} - \bar{y} - D^e \bar{d}, \\ \dot{\bar{d}} &= \bar{S}\bar{d}, \\ \dot{\bar{z}} &= L\bar{\lambda},\end{aligned}$$

where $\bar{\lambda} = \text{col}(\bar{\lambda}_1, \dots, \bar{\lambda}_N), \bar{z} = \text{col}(\bar{z}_1, \dots, \bar{z}_N), \bar{y} = \text{col}(\bar{y}_1, \dots, \bar{y}_N), \bar{d} = \text{col}(\bar{d}_1, \dots, \bar{d}_N), \bar{S} = \text{diag}\{S_1 - L_1 D_1, \dots, S_N - L_N D_N\}$ and $D^e = \text{diag}\{D_1^e, \dots, D_N^e\}$.

Letting $\hat{z}_1 = r^T \bar{z}, \hat{z}_2 = R^T \bar{z}$ gives

$$\begin{aligned}\dot{\bar{\lambda}} &= -L\bar{\lambda} - LR\hat{z}_2 - \bar{y} - D^e \bar{d}, \\ \dot{\hat{z}}_2 &= R^T L\bar{\lambda}, \\ \dot{\bar{d}} &= \bar{S}\bar{d},\end{aligned}\tag{12}$$

where we use $r^T \bar{z} = 0$ and $\hat{z}_1 \equiv 0$. Then our problem is reduced to prove the exponential stability of the composite system determined by the evolution of $\bar{\lambda}, \bar{d}, \hat{z}_2$ under (5), (9), and (12).

The proof will be accomplished by two steps.

First, we prove the exponential stability of (12) when $\bar{y} \equiv 0$. Noticing that the system is in a cascaded form, we only have to prove the stability of the $(\bar{\lambda}, \hat{z}_2)$ -subsystem when $\bar{y} \equiv 0$ and $\bar{d} \equiv 0$, since \bar{S} is already Hurwitz by the selection of L_i .

Since system (12) is linear, we only have to obtain the asymptotic stability of $(\bar{\lambda}, \hat{z}_2)$ -subsystem. For this purpose, we consider $V_{Iz} = \frac{1}{2} \bar{\lambda}^T \bar{\lambda} + (1/2) \bar{z}_2^T \bar{z}_2$ as a Lyapunov candidate, and then its derivative along the trajectory of (12) when $\bar{y} \equiv 0$ and $\bar{d} \equiv 0$ satisfy the following:

$$\dot{V}_{Iz} \leq -\bar{\lambda}^T L \bar{\lambda} \leq -c \bar{\lambda}^T \bar{\lambda},$$

where c is the minimal positive eigenvalue of L . This consequently implies $\bar{\lambda} \rightarrow 0$ as t goes to infinity. Denote

$$\bar{A} \triangleq \begin{bmatrix} -L & LR \\ R^T L & 0 \end{bmatrix}.$$

Under Assumption 2, LR has a full column-rank, then the pair $([I \ 0], \bar{A})$ is observable by the PBH-test. Combining the above

arguments, we can conclude the asymptotic stability of $(\bar{\lambda}, \hat{z}_2)$ -subsystem when $\bar{y} \equiv 0$ and $\bar{d} \equiv 0$ and thus the exponential stability of (12) when $\bar{y} \equiv 0$.

Next, we prove the exponential stability of the composite system. Since (12) is exponentially stable when $\bar{y} \equiv 0$, there exists a unique positive definite matrix P satisfying $\hat{A}^T P + P \hat{A} = -I$ for $\hat{A} \triangleq \text{diag}\{\bar{A}, \bar{S}\}$. Take a Lyapunov candidate for the composite system as $\bar{V} = \sum_{i=1}^N V_i(x_i, x_i^*) + \hat{\lambda}^T P \hat{\lambda}$ with $\hat{\lambda} = \text{col}(\bar{\lambda}, \hat{z}_2, \bar{d})$, which is apparently positive definite due the exponential passivity of (5) by assumptions. Its derivative along the trajectory of this composite system composed of (5) and (9) satisfies

$$\begin{aligned}\dot{\bar{V}} &\leq - \sum_{i=1}^N c_{ii} V_i - \sum_{i=1}^N \bar{y}_i^T [\hat{u}_i(y_i) - \hat{u}_i(y_i^*)] + \bar{y}^T \bar{\lambda} \\ &\quad - \hat{\lambda}^T \hat{\lambda} + 2\hat{\lambda}^T P \bar{A} \bar{B} \bar{y},\end{aligned}$$

where $\bar{B} = \text{col}(I_N, \mathbf{0}, \mathbf{0})$. Using Young's inequality and $\|\bar{\lambda}\|^2 \leq \|\hat{\lambda}\|^2$, one can obtain

$$\begin{aligned}\dot{\bar{V}} &\leq - \sum_{i=1}^N c_{ii} V_i - \sum_{i=1}^N \bar{y}_i^T [\hat{u}_i(y_i) - \hat{u}_i(y_i^*)] + \|\bar{y}\|^2 \\ &\quad + \frac{1}{4} \|\bar{\lambda}\|^2 - \hat{\lambda}^T \hat{\lambda} + \frac{1}{4} \|\hat{\lambda}\|^2 + 4\|P\bar{A}\bar{B}\|^2 \|\bar{y}\|^2 \\ &= - \sum_{i=1}^N c_{ii} V_i - \frac{1}{2} \hat{\lambda}^T \hat{\lambda} + (1 + 4\|P\bar{A}\bar{B}\|^2) \|\bar{y}\|^2 \\ &\quad + \sum_{i=1}^N \bar{y}_i^T [\hat{u}_i(y_i) - \hat{u}_i(y_i^*)].\end{aligned}$$

Taking

$$\gamma > \max_i \left(\frac{2 + 4\|P\bar{A}\bar{B}\|^2 + M_i}{\underline{h}_i} \right)$$

gives

$$\dot{\bar{V}} \leq - \sum_{i=1}^N c_{ii} V_i - \frac{1}{2} \|\hat{\lambda}\|^2 - \|\bar{y}\|^2.$$

Applying Theorem 4.10 in [30] gives the exponential convergence of \bar{V} under this algorithm and thus y exponentially converges to the optimal solution of (6). The proof is complete. \square

Remark 3: In contrast to the existing constrained steady-state regulation problem [24], we consider its distributed extensions where the steady-state of agents can only be determined and reached in a distributed way, which is of course more challenging. Moreover, unknown observation disturbances are taken into consideration, along with both the asymptotic and exponential convergence results, while only local and asymptotic results were obtained in [10, 24].

5 Applications and discussions

In this section, we provide the applications of previous designs and examples to verify the effectiveness.

5.1 Distributed inventory control

In this subsection, we show how a distributed inventory control problem can be formulated as a resource allocation problem over dynamic agents and solved by our approach. We consider only one perishable commodity and N networked inventories [25].

The inventory system at node i is modelled as

$$\dot{I}_i = -\theta_i I_i + P_i - D_i,\tag{13}$$

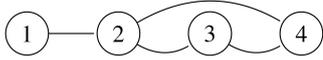


Fig. 1 Communication graph \mathcal{G}

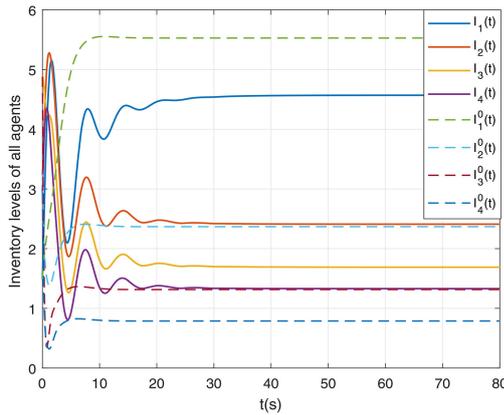


Fig. 2 Profiles of the inventory levels under the control (15)

where I_i is the inventory level, $\theta_i > 0$ is the deterioration rate, P_i is the production rate at node i , and D_i is a constant demand rate. The information structure among these inventories is represented by a connected graph \mathcal{G} . The storage cost at each warehouse is given as $f_i(I_i) = \alpha_i I_i^2 + \beta_i I_i + \gamma_i$, where $\alpha_i > 0$.

Generally speaking, we aim to maintain the total inventory at a certain level I^* to satisfy the customer's demands and some safety goals. Thus, this inventory control problem can be formulated as follows. Given inventory systems and cost functions $f_1(\cdot), \dots, f_N(\cdot)$, find a production rate for each inventory in a distributed way, such that the inventory level I converges to the optimal solution $I^* \triangleq \text{col}(I_1^*, \dots, I_N^*)$ that solves

$$\begin{aligned} & \text{minimize} && \sum_{i=1}^N f_i(I_i) \\ & \text{subject to} && \sum_{i=1}^N I_i = I^*. \end{aligned} \quad (14)$$

Clearly, the i th inventory system is exponentially passive w.r.t. $(I_i^*, \mathbf{u}_i(I_i^*))$ with input $y = I_i$, input P_i and $\mathbf{u}_i(I_i^*) = \theta_i I_i^* + D_i$, and hence Assumptions 1–4 are also satisfied. With a pre-allocation of inventory level $I^* = \sum_{i=1}^N I_i^*$, the following corollary shows the effectiveness of our previous design on distributed inventory control.

Corollary 1: Given the communication graph \mathcal{G} and cost functions $f_1(\cdot), \dots, f_N(\cdot)$, the distributed inventory control problem determined by (13) and (14) can be solved by the following algorithm

$$\begin{aligned} P_i &= -\nabla f_i(I_i) + \lambda_i + \mathbf{u}_i(I_i), \\ \dot{\lambda}_i &= -\lambda_i^\nu - z_i^\nu + I_i^r - I_i, \\ \dot{z}_i &= \lambda_i^\nu, \quad i = 1, \dots, N, \end{aligned} \quad (15)$$

where γ can be any positive constant. Moreover, $I(t)$ converges to I^* exponentially as $t \rightarrow \infty$.

We then provide a numerical example with four inventories having parameters $\alpha_i = 0.1i$, $\beta_i = -0.05i$, $\gamma_i = \theta_i = D_i = I_i^* = i$, $i = 1, \dots, 4$. The communication graph is taken as Fig. 1 and all initial conditions are randomly chosen in $[0, 6]$. By choosing control inputs as (15), we solve this inventory control problem and drive the outputs to the optimal solution $I^* = \text{col}(4.57, 2.41, 1.69, 1.33)$. For comparisons, we take the input P_i as that in [22], and the output trajectories of inventories are represented by dash lines. It can be found in Fig. 2 that the

algorithm fails to solve our problem and only drives the outputs of agents to a non-optimal point $\text{col}(5.53, 2.37, 1.32, 0.79)$, which confirms the effectiveness of our design.

5.2 Average consensus with disturbance rejection

A consensus and especially an average consensus of multi-agent agents have been shown as an inevitable part of the solution for more complex problems in several applications, including distributed filtering and multi-robot flocking [1, 2]. While a consensus only requires the agreement on some common signals, an extra condition has to be satisfied in an average consensus, which relates the limiting behaviour of the whole system to the initial states. The average consensus problem is certainly more challenging especially when we expect an average consensus of all outputs for a heterogeneous multi-agent network.

In our formulation, let $f_i(s) = (1/2)s^2$ and one can obtain the following conclusion.

Corollary 2: Under Assumptions 2–4, the outputs of agents (5) can reach the average of their private data d_i^0 under the following algorithm

$$\begin{aligned} u_i &= -\gamma y_i + \lambda_i + \mathbf{u}_i(y_i), \\ \dot{\lambda}_i &= -\lambda_i^\nu - z_i^\nu + d_i - y_i - D_i^e \eta_i, \\ \dot{\eta}_i &= (S_i - L_i D_i) \eta_i + L_i d_i, \\ \dot{z}_i &= \lambda_i^\nu, \quad i = 1, \dots, N, \end{aligned} \quad (16)$$

where $\gamma > 1 + \max_i M_i$ and L_i is selected as in Theorem 1, i.e. $\lim_{t \rightarrow +\infty} y_i(t) = \frac{1}{N} \sum_{i=1}^N d_i^0$ for $i = 1, \dots, N$.

The proof is a direct application of Theorem 1. As a by-product of this corollary, we can solve the output average consensus problem of these agents in spite of observation disturbances d_i^e by letting $d_i^0 = y_i(0)$. Then, $y_i(t) \rightarrow \text{Aver}(y(0)) \triangleq (1/N) \sum_{i=1}^N y_i(0)$ as t goes to infinity. Since a single integrator is passive, this result extends the average consensus results to a larger class of non-linear systems with disturbance rejection.

To verify the effectiveness of this algorithm, we consider four controlled Chua's circuits [42] as follows:

$$\begin{aligned} \dot{x}_{i1} &= \alpha_i(x_{i2} - x_{i1} - f_i(x_{i1}) + F_i), \\ \dot{x}_{i2} &= x_{i1} - x_{i2} + x_{i3}, \\ \dot{x}_{i3} &= -\beta_i x_{i2}, \\ y_i &= x_{i1}, \end{aligned}$$

where F_i is the input signal and $f_i(x_{i1}) = b_i x_{i1} + (1/2)(a_i - b_i)(|x_{i1} + c_i| - |x_{i1} - c_i|)$ with typical parameters $\alpha_i = 9$, $\beta_i = 100/7$, $a_i = -(8/7)$, $b_i = -(5/7)$, $c_i = 1$. The agents are coupled by a communication graph as shown in Fig. 1. We aim to achieve an output average consensus by output feedback control.

First, we let $F_i = u_i + f_i(x_{i1})$ to passivate agent i 's dynamics with output y_i and a new input u_i . In fact, Assumptions 3 and 4 hold with $\mathbf{x}_{i1}(r) = r$, $\mathbf{x}_{i2}(r) = 0$, $\mathbf{x}_{i3}(r) = -r$, $\mathbf{u}_i(r) = r$, and $V_i(x, x_i^*) = (1/\alpha_i)(x_{i1} - x_{i1}^*)^2 + (x_{i2} - x_{i2}^*)^2 + (1/\beta_i)(x_{i3} - x_{i3}^*)$. Thus the average output consensus problem among these agents can be solved by (16).

For simulations, we assume agent i is subject to a sinusoidal observation disturbance with frequency $\omega_i = 4 - i$ and unknown amplitude or phase after $t = 30$ s. The controller is taken as (9) with $D_i^e = [0, 0]$ during $0 \text{ s} \sim 45 \text{ s}$ and with $D_i^e = [1, 0]$ after 45 s. All initial conditions are randomly chosen in $[-5, 5]$. At first, the outputs of agents quickly converge to their average point. Then, the average consensus is disrupted by those observation disturbances. After the disturbance rejection part works at $t = 45$ s, we recover the output average consensus of these agents. The detailed performance of the above control is depicted in Fig. 3.

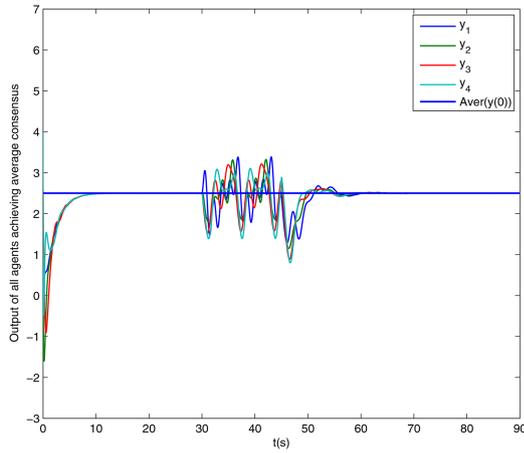


Fig. 3 Profiles of the outputs achieving an average consensus

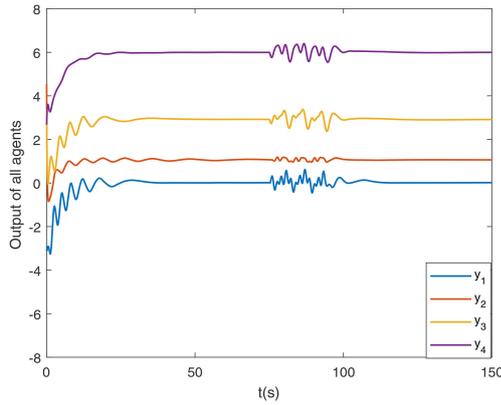


Fig. 4 Profiles of the outputs y_i under our controller (9)

5.3 Non-minimum phase multi-agent coordination

Note that many non-minimum phase non-linear systems have an incremental passivity property (perhaps after a passivation procedure) [30]. It is appealing to employ the proposed algorithms to handle non-minimum phase non-linear agents with more complicated objective functions. We present an example in this subsection to verify this point.

Consider a network of four non-linear agents described by

$$\begin{aligned}\dot{z}_{i1} &= \varepsilon_{i1} z_{i2}^3, \\ \dot{z}_{i2} &= -\varepsilon_{i2} z_{i1} + \varepsilon_{i3} x_i, \\ \dot{x}_i &= -\varepsilon_{i4} z_{i2}^3 - \varepsilon_{i5} x_i + u_i, \\ y_i &= x_i,\end{aligned}$$

where $\varepsilon_{i1}, \dots, \varepsilon_{i5}$ are positive constants ($i = 1, \dots, 4$).

Apparently, the zero dynamics of agent i is $\dot{z}_{i1} = \varepsilon_{i1} z_{i2}^3$, $\dot{z}_{i2} = -\varepsilon_{i2} z_{i1}$, which is not asymptotically stable. Thus, the agents are all in the non-minimum phase. Nevertheless, it can be verified that Assumptions 3 and 4 hold with $\mathbf{z}_{i1}(r) = (\varepsilon_3/\varepsilon_2)r$, $\mathbf{z}_{i2}(r) = 0$, $\mathbf{x}_i(r) = r$, $\mathbf{u}_i(r) = \varepsilon_{i5}r$ and storage functions

$$V_i = \frac{1}{2} z_{i1}^2 + \frac{\varepsilon_{i1}}{4\varepsilon_{i3}} z_{i2}^4 + \frac{\varepsilon_{i1}\varepsilon_{i3}}{2\varepsilon_{i2}\varepsilon_{i4}} x_i^2.$$

Take $\varepsilon_{ij} = 1$ and an information sharing graph as in Fig. 1. We consider the distributed coordination problem among these agents with regulation constraints. The local cost functions satisfying Assumption 1 are chosen as $f_1(y_1) = (y_1 + 3)^2$, $f_2(y_2) = y_2^2 \ln(1 + y_2^2) + (y_2 + 1)^2$, $f_3(y_3) = \ln(e^{-0.1y_3} + e^{0.3y_3}) + y_3^2$ and $f_4(y_4) = y_4^2 / (25\sqrt{y_4^2 + 1}) + (y_4 - 3)^2$. Assume the constant $d_i^0 = i$ and agent i is subject to a sinusoidal disturbance with frequency $\omega_i = 4 - i$ but

unknown amplitude or phase after $t = 75$ s. The problem is solvable by Theorem 1.

Choose $L_1 = \text{col}(5.00, 6.72, 2.19)$, $L_2 = \text{col}(5.00, 6.51, 2.75)$, $L_3 = \text{col}(5.00, 6.07, 3.69)$, $L_4 = \text{col}(5.00, 5.00, 5.00)$ and $\gamma = 2$. To verify the disturbance rejection performance, we let $D_i^e = [0, 0]$ during $0 \text{ s} \sim 95 \text{ s}$ and $D_i^e = [1, 0]$ after 95 s . The evolution of y_i under (9) is depicted in Fig. 4.

At first, all outputs of agents evolve without disturbances and quickly converge the optimal point. Then, the agents are moved away from the optimal steady-state due to the observation disturbances. After the disturbance rejection part works at $t = 95 \text{ s}$, we recover the optimal steady-state regulation of these agents, which confirms the conclusions.

6 Conclusions

A distributed coordination problem with regulation constraints was formulated and solved for a class of non-linear passive multi-agent systems. By reviewing the passivity technique w.r.t. non-zero equilibrium, we reduce the concerned optimisation to a passivity-based regulation problem. Combined with graph theory and observer design technique, gradient-based rules are proposed to solve our problem with disturbance rejection. Potential applications and numerical examples are presented to show their effectiveness. In fact, many interesting and challenging problems still remain to be addressed, including how to solve this problem under switching graphs and extend the gradient-based rules to general monotone-operator-based designs.

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8 References

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