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Relative attitude formation control of multi-agent systems

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SUMMARY

This paper investigates the relative attitude formation control problem for a group of rigid-body agents using relative attitude information on $SO(3)$. On the basis of the gradient of a potential function, a family of distributed angular velocity control laws, which differ in the sense of a geodesic distance dependent function, is proposed. With directed and switching interaction topologies, the desired relative attitude formation is showed to be achieved asymptotically provided that the topology is jointly quasi-strongly connected. Moreover, several sufficient conditions for the desired formation to be achieved exponentially and almost globally are given. Additionally, numerical examples are provided to illustrate the effectiveness of the proposed distributed control laws. Copyright © 2017 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Recently, consensus or synchronization of multi-agent systems has been widely studied in order to make all agents reach a common state of interest. Attitude synchronization, in fact, is a nonlinear consensus problem defined on the Lie group $SO(3)$, which consists of all orthogonal matrices in $\mathbb{R}^{3\times 3}$ with unit determinant, and it has also been a hot topic with many applications in practical systems including satellites, spacecrafts, and mobile robots [1–5].

In real applications, attitude formation is an important practical problem, especially for multi-camera coordination [6] and formation flying [7, 8], which is a generalization of attitude synchronization. Different from the widely studied attitude synchronization [1–5], attitude formation forces the relative attitudes between agents to achieve the desired ones. Because of the inherent nonlinearity, very few results on attitude formation are obtained in general situations. In fact, a spacecraft formation problem addressed in [9–11] forces a group of agents to track a common time-varying reference attitude (which might be viewed as a leader), which is converted to a leader–follower problem to synchronize the orientations of the agents eventually.

It is well known that the connectivity of interaction topologies is a key to achieving the collective behavior in a multi-agent network. Existing results on attitude coordination mostly require the interaction topology to be connected at every time [1, 3, 4, 6, 8–13]. However, multi-agent interaction topologies may change over time in practice because of link failures and energy saving. In the study of such variable topologies, a well-known connectivity assumption, joint connection, is employed to guarantee multi-agent consensus for first-order or second-order linear or nonlinear systems [2, 5, 14, 15] without requiring connectedness of the graph at every moment.

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The objective of this paper is to study the distributed attitude formation control design on the basis of relative attitude information on $SO(3)$. Here, we formulate the desired formation using relative attitudes between agents on $SO(3)$ directly and then propose a family of distributed control schemes employing the three parameter representations of error attitudes. With switching and directed interaction topologies, we show that the desired relative attitude formation can be achieved when the topology is jointly quasi-strongly connected (JQSC). We also give an estimation for the region of convergence from which the relative attitude between any two agents is asymptotically stabilized to the desired value. Moreover, we provide several sufficient conditions for the relative attitude formation to be achieved exponentially and almost globally.

Existing results on attitude formation mostly use absolute attitudes of the agents to define the desired formation [16]. In [12, 13], the formation is formulated on the difference of the Modified Rodriguez Parameters for the orientations of any two agents, and a leader as the role of a reference attitude is defined to make the formation meaningful on $SO(3)$, where the absolute attitude of each agent in these papers actually is known or could be computed beforehand. Unlike these formulations, we focus on the desired relative attitude directly on the attitude space $SO(3)$ and give the distributed control on the basis of neighbor information without requiring a reference leader. In [8], the desired absolute attitude of any agent is not assigned beforehand also; however, the underlying interaction topology is assumed to be an undirected ring.

Concerning a system of rigid-body agents, it is common to assume the absolute attitude of each agent to a global inertial frame is available and the agents can transmit their absolute attitudes to other agents [1, 3, 9–13, 16]. However, a global reference frame is not always available. Even when it is available, for example, using GPS, interference from the measurement process may influence the accuracy of the final formation. In this paper, the agents are only able to measure the relative orientations to their neighbors on $SO(3)$, whereas the absolute attitudes of agents are not required in the control. Furthermore, the proposed control scheme for attitude formation problem here is also based on the well-known joint connection assumption for directed and switching interaction graphs.

This paper is organized as follows. In Section 2, necessary preliminaries for the rotational motion on $SO(3)$ and graph theory are given. In Section 3, relative attitude formation between the agents is formulated, and properties of error attitudes are presented. In Section 4, the proposed control scheme is introduced, and then the convergent result with directed and switching topologies is given. Following that, the exponential convergence and global behaviors of the proposed control with fixed inter-agent graph are discussed in Section 5. Finally, illustrative examples are provided in Section 6, and the paper is concluded in Section 7.

2. PRELIMINARIES

In this section, we give preliminaries on three-dimensional attitudes and graph theory for the following analysis in this paper. We first introduce the *hat* and *vee* operators used throughout the paper. Let $so(3)$ be the set of skew-symmetric matrices in $\mathbb{R}^{3\times 3}$. The hat operator is a map $\wedge : \mathbb{R}^3 \to so(3)$ transforming a vector $\boldsymbol{a} = [a_1, a_2, a_3]^T$ to $\widehat{\boldsymbol{a}}$ in the form of [17, 18]

$$\widehat{\boldsymbol{a}} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$$

It holds that $\widehat{\boldsymbol{a}}\boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{b}$ for any $\boldsymbol{b} \in \mathbb{R}^3$. The inverse of the hat map, denoted by the vee operator $\vee : so(3) \to \mathbb{R}^3$, extracts the components of vector $\boldsymbol{a}$ from $\widehat{\boldsymbol{a}}$. Some useful identities related to these two operators are summarized as follows:

$$\widehat{\boldsymbol{a}}^2 = -\boldsymbol{a}^T\boldsymbol{a}I_3 + \boldsymbol{a}\boldsymbol{a}^T, \tag{1}$$

$$\mathrm{tr}(\widehat{\mathbf{a}}\widehat{\mathbf{b}}) = -2\boldsymbol{a}^T\boldsymbol{b}, \tag{2}$$

$$(\widehat{\boldsymbol{a}}A + A^T\widehat{\boldsymbol{a}})^\vee = (\mathrm{tr}(A)I_3 - A)\boldsymbol{a}, \tag{3}$$

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$$\mathrm{tr}(A\widehat{\boldsymbol{a}}) = -\boldsymbol{a}^T(A - A^T)^\vee, \tag{4}$$

where $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$, $A \in \mathbb{R}^{3\times 3}$ and $\mathrm{tr}(\cdot)$ stands for the matrix trace.

2.1. Three-dimensional attitudes

In this paper, we consider the pure rotational motion of rigid bodies. Let $\mathcal{A}$ be a reference frame and $\mathcal{B}$ be the body-fixed frame attached on a rigid body. The attitude of the rigid body relative to the reference fame $\mathcal{A}$ is defined by the rotational transformation matrix from $\mathcal{A}$ to $\mathcal{B}$, denoted by $R_{ab} \in \mathbb{R}^{3\times 3}$. It is known that the set of attitudes forms a Lie group, denoted by $SO(3)$ [19], that is $SO(3) = \{R \in \mathbb{R}^{3\times 3} \mid R^T R = I_3, \det(R) = 1\}$. From the transport theorem [20], the kinematics of the relative attitude R_{ab} is governed by

$$\dot{R}_{ab} = \widehat{\boldsymbol{\omega}}^{\mathcal{A}}_{ab} R_{ab} = R_{ab}\widehat{\boldsymbol{\omega}}^{\mathcal{B}}_{ab}, \tag{5}$$

where $\boldsymbol{\omega}^{\mathcal{A}}_{ab}$ is the angular velocity of the rigid body viewed from the reference frame $\mathcal{A}$ and resolved in $\mathcal{A}$ as well and $\boldsymbol{\omega}^{\mathcal{B}}_{ab}$ is the same physical velocity resolved in $\mathcal{B}$. It holds that $\boldsymbol{\omega}^{\mathcal{A}}_{ab} = R_{ab}\boldsymbol{\omega}^{\mathcal{B}}_{ab}$.

Remark 2.1
The attitude in this paper follows the definition given in [17, 18] determined by the rotational transformation from the reference frame to the body-fixed frame. In some other references (e.g.,[21, 22]), the relative attitude is defined by the rotational transformation between different coordinate expressions of the same physical vector in different frames (from $\mathcal{A}$ to $\mathcal{B}$), which is, in fact, the inverse of the transformation in our case.

For any $R \in SO(3)$ and $\boldsymbol{a} \in \mathbb{R}^3$, the following equality holds

$$R\widehat{\boldsymbol{a}}R^T = (R\boldsymbol{a})^\wedge. \tag{6}$$

For a rotation $R \in SO(3)$, its angle and axis $(\theta, \boldsymbol{k})$ is determined using

$$\theta = \arccos\left((\mathrm{tr}(R) - 1)/2\right) \in [0, \pi], \tag{7}$$

$$\boldsymbol{k} = \frac{(R - R^T)^\vee}{2\sin\theta} \in \mathcal{S}^2, \tag{8}$$

where $\mathcal{S}^2 = \{\boldsymbol{u} \in \mathbb{R}^3 \mid \boldsymbol{u}^T\boldsymbol{u} = 1\}$ is the two-sphere. Notice that the equation (8) for the rotation axis $\boldsymbol{k}$ is valid only when $\theta \in (0, \pi)$. Indeed, $\boldsymbol{k}$ is not determined when $\theta = 0$, and there are two axes with an opposite sign each other when $\theta = \pi$. On the contrary, the rotation R corresponding to any angle $\theta \in [0, \pi]$ and axis $\boldsymbol{k} \in \mathcal{S}^2$ is given by the Rodrigues' formula [19]

$$R = \exp(\theta\widehat{\boldsymbol{k}}) = I_3 + \widehat{\boldsymbol{k}}\sin\theta + \widehat{\boldsymbol{k}}^2(1 - \cos\theta), \tag{9}$$

where $\exp(\cdot)$ is the matrix exponential. From (7) and (9), we have that $R = I_3$ if and only if $\theta = 0$.

The angle of a rotation on $SO(3)$ has a geometric interpretation as the Riemannian distance. Namely, define the metric tensor $\langle\cdot,\cdot\rangle_R$ for any rotation $R \in SO(3)$ as

$$\langle V_1, V_2\rangle_R = \frac{1}{2}\mathrm{tr}(V_1^T V_2), \quad \forall V_1, V_2 \in T_R SO(3), \tag{10}$$

where $T_R SO(3) = \{R\widehat{\boldsymbol{v}} \mid \boldsymbol{v} \in \mathbb{R}^3\} = \{\widehat{\boldsymbol{v}}R \mid \boldsymbol{v} \in \mathbb{R}^3\}$ is the tangent space of $SO(3)$ at R. Then the Riemannian distance between any two rotations $P, Q \in SO(3)$, which also is the length of the shortest geodesic curve connecting the two rotations, equals the angle of their relative rotation [19], that is,

$$d_{SO(3)}(P, Q) = \theta(P^T Q) \in [0, \pi].$$

A set $B_r(Q) \subset SO(3)$ centered at a rotation $Q \in SO(3)$ is called an open (geodesic) ball of radius r if $B_r(Q) = \{R \in SO(3) \mid d_{SO(3)}(R, Q) < r\}$.

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Let $f : SO(3) \to \mathbb{R}$ be a continuous function. For any $R \in SO(3)$, if there exists a unique tangent vector in $T_R SO(3)$, denoted by $\nabla_R f$, such that

$$\frac{d}{ds} f(\gamma(s))\bigg|_{s=0} = \langle \nabla_R f, V \rangle_R, \quad \forall V \in T_R SO(3),$$

where $\gamma(s)$ is any curve on $SO(3)$ passing $\gamma(0) = R$ with tangent vector $\gamma'(0) = V$, then the gradient of f on $SO(3)$ at R exists and equals $\nabla_R f$ (see [23] for details). Suppose $R_{ab}(t)$ is a smooth trajectory of (5) and $\nabla_{R_{ab}(t)} f$ exists at time t, then $f(R_{ab}(t))$ is differentiable at t and can be computed by the following equation

$$\dot{f}(R_{ab}) = \langle \nabla_{R_{ab}} f, \dot{R}_{ab} \rangle_{R_{ab}}. \tag{11}$$

The following two lemmas are useful for the properties of rotations on $SO(3)$.

Lemma 2.2
Suppose $f(P) = h(d_{SO(3)}(P, Q))$, where $Q \in SO(3)$ and $h : [0, \pi] \to \mathbb{R}$ is continuous differentiable. The gradient of f on $SO(3)$ at $P \in SO(3)$ is given as

$$\nabla_P f = \begin{cases} -P\, h'(\theta)\, \widehat{\boldsymbol{k}}, & \text{if } \theta \in (0, \pi) \\ O_3, & \text{if} \theta = 0 \text{ or } \pi, \text{and } h'(\theta) = 0 \\ \text{not exist}, & \text{if } \theta = 0 \text{ or } \pi, \text{and } h'(\theta) \neq 0 \end{cases},$$

where $(\theta, \boldsymbol{k})$ is the angle and axis of the relative rotation $P^T Q$.

Lemma 2.3
For any $P, Q \in SO(3)$ with $(\theta_p, \boldsymbol{k}_p)$ and $(\theta_q, \boldsymbol{k}_q)$ as their respective angles and axes, the following equations hold for the angle and axis $(\theta, \boldsymbol{k})$ of the relative rotation $P^T Q$:

$$\begin{aligned} \cos\frac{\theta}{2} &= \left|\cos\frac{\theta_p}{2}\cos\frac{\theta_q}{2} + \sin\frac{\theta_p}{2}\sin\frac{\theta_q}{2}\boldsymbol{k}_p^T\boldsymbol{k}_q\right|, \\ \cos\frac{\theta}{2}\sin\frac{\theta}{2}\boldsymbol{k} &= \left(\cos\frac{\theta_p}{2}\cos\frac{\theta_q}{2} + \sin\frac{\theta_p}{2}\sin\frac{\theta_q}{2}\boldsymbol{k}_p^T\boldsymbol{k}_q\right) \\ &\quad \left(-\sin\frac{\theta_p}{2}\cos\frac{\theta_q}{2}\boldsymbol{k}_p + \cos\frac{\theta_p}{2}\sin\frac{\theta_q}{2}\boldsymbol{k}_q - \sin\frac{\theta_p}{2}\sin\frac{\theta_q}{2}\widehat{\boldsymbol{k}}_p\boldsymbol{k}_q\right). \end{aligned}$$

Lemma 2.2 can be showed by taking $\gamma(s) = P\exp(\widehat{\boldsymbol{v}}s)$, $\boldsymbol{v} \in \mathbb{R}^3$ and then computing $(df(\gamma(s))/ds)|_{s=0}$. And Lemma 2.3 can be verified by expanding the identities $1 + 2\cos\theta = \mathrm{tr}(P^T Q)$ and $2\boldsymbol{k}\sin\theta = P^T Q - Q^T P$ with the Rodrigues' formula (9) of P and Q. The details of the proofs are omitted here.

2.2. *Graph theory*

A directed graph (digraph) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a node set $\mathcal{V} = \{1, 2, \ldots, n\}$ and an edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, in which an edge is an ordered pair of distinct nodes. A node j is said to be a neighbor of i if $(j, i) \in \mathcal{E}$, and $\mathcal{N}_i = \{j \mid (j, i) \in \mathcal{E}\}$ is denoted as the set of neighbors of node i. A directed path of $\mathcal{G}$ is a sequence of distinct nodes in $\mathcal{V}$ such that any consecutive nodes in the sequence correspond to an edge of $\mathcal{G}$. A node j is said to be connected to i if there is a directed path from j to i, and j is a root if it is connected to every other node. A digraph $\mathcal{G}$ is said to be quasi-strongly connected (QSC) if $\mathcal{G}$ contains a root node. $\mathcal{G} = \cup_{i=1}^m \mathcal{G}_i$ is referred to as the union of a collection of digraphs $\{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_m\}$, each with vertex set $\mathcal{V}$, if $\mathcal{G}$ has the node set $\mathcal{V}$ and edge set $\mathcal{E}$ equaling the union of the edge sets of all the digraphs in the collection.

The adjacency matrix A of a digraph $\mathcal{G}$ is defined such that $a_{ij} = 1$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. Then the Laplacian $L = [l_{ij}]_{n \times n}$ of $\mathcal{G}$ is defined such that $l_{ij} = -a_{ij}$ when $i \neq j$ and $l_{ij} = \sum_{k=1}^n a_{ik}$ when $i = j$. A digraph $\mathcal{G}$ is QSC if and only if its Laplacian has a simple zero eigenvalue and all other eigenvalues have positive real parts.

A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is undirected if the node pair of each edge is unordered. An undirected graph is connected if there is an undirected path between every pair of distinct nodes. A connected component of $\mathcal{G}$ is a maximal connected subgraph of $\mathcal{G}$. An undirected graph with n nodes is a tree if it is connected and has $n-1$ edges. The nodes with only one neighbor in a tree are called leaves. The induced subgraph of $\mathcal{G}$ by a nonempty node set $\mathcal{U} \subset \mathcal{V}$ is $\mathcal{G}_{\mathcal{U}} = (\mathcal{U}, \mathcal{E} \cap (\mathcal{U} \times \mathcal{U}))$. The details can be found in [24, 25].

3. PROBLEM FORMULATION

We investigate the relative attitude formation control problem for a group of n ($n \geqslant 3$) rigid-body agents. In this section, we formulate the problem and give the definition and properties of the error attitudes of the formation.

3.1. *Relative attitude formation*

Let $\mathcal{V} = \{1, 2, \ldots, n\}$ represent the set of agents, and suppose an inertial reference frame $\mathcal{F}$ and the body-fixed frame $\mathcal{B}_i$ for agent $i \in \mathcal{V}$ are defined. We denote R_i and R_{ij} as the attitudes of $\mathcal{B}_i$ relative to $\mathcal{F}$ and $\mathcal{B}_j$ relative to $\mathcal{B}_i$, respectively. Then it holds that $R_{ij} = R_i^T R_j$. We regard R_i as the *absolute attitude* of agent i and R_{ij} as the *relative attitude* of agent j to agent i. Let $\boldsymbol{\omega}_i$ be the angular velocity of agent i with respect to the inertial frame $\mathcal{F}$ resolved in $\mathcal{B}_i$, which is the control variable in our problem. Moreover, we denote $\boldsymbol{\omega}_{ij}$ as the angular velocity of agent j viewed from agent i and resolved in $\mathcal{B}_i$, that is, $\boldsymbol{\omega}_{ij} = R_{ij}\boldsymbol{\omega}_j - \boldsymbol{\omega}_i$. Similarly, we refer to $\boldsymbol{\omega}_i$ as *absolute angular velocity* and $\boldsymbol{\omega}_{ij}$ as *relative angular velocity*. From (5), the respective kinematics of R_i and R_{ij} are given as

$$\dot{R}_i = R_i \widehat{\boldsymbol{\omega}}_i, \quad \dot{R}_{ij} = \widehat{\boldsymbol{\omega}}_{ij} R_{ij}. \tag{12}$$

In the paper, a desired relative attitude formation between the agents is expected to achieve, denoted as

$$\left\{ R_{ij}^d(t) : [0, \infty) \to SO(3) \,|\, i, j \in \mathcal{V}, i \neq j \right\}, \tag{13}$$

where $R_{ij}^d(t)$ is a smooth function of time representing the desired relative attitude of agent j to agent i. In the following, whenever there is no confusion, we drop the explicit dependence of R_{ij}^d on time. To achieve a feasible formation for the group of rigid-body agents, the desired formation (13) should be compatible, that is,

$$R_{ij}^d R_{ji}^d = I_3, \quad R_{ik}^d = R_{ij}^d R_{jk}^d, \quad \forall i, j, k \in \mathcal{V}. \tag{14}$$

Suppose that, for any possible neighbor agent j of agent i, the desired relative attitude R_{ij}^d is known to agent i.

Because R_{ij}^d evolves on $SO(3)$, $\dot{R}_{ij}^d \in T_{R_{ij}^d} SO(3)$. Hence, $\dot{R}_{ij}^d R_{ji}^d \in so(3)$, and we denote $\boldsymbol{\omega}_{ij}^d = (\dot{R}_{ij}^d R_{ji}^d)^\vee$ as the desired relative angular velocity of agent j to agent i resolved in $\mathcal{B}_i$. Then R_{ij}^d satisfies the following differential equation

$$\dot{R}_{ij}^d = \widehat{\boldsymbol{\omega}}_{ij}^d R_{ij}^d. \tag{15}$$

For each agent $i \in \mathcal{V}$, let $\boldsymbol{\omega}_i^d(t) : [0, \infty) \to \mathbb{R}^3$ be a smooth function of time. Then $\boldsymbol{\omega}^d(t) = \{\boldsymbol{\omega}_i^d(t)\}_{i \in \mathcal{V}}$ can be viewed as the desired absolute angular velocity for the system of agents if the following equations hold

$$\boldsymbol{\omega}_{ij}^d(t) = R_{ij}^d(t)\boldsymbol{\omega}_j^d(t) - \boldsymbol{\omega}_i^d(t), \quad \forall i, j \in \mathcal{V}, \ t \geqslant 0. \tag{16}$$

In the paper, we select an arbitrary function $\boldsymbol{\omega}^d(t)$ satisfying (16) and assume that $\boldsymbol{\omega}_i^d(t)$ is known to agent i. In fact, we can choose $\boldsymbol{\omega}_1^d(t) = -\boldsymbol{\omega}_{12}^d(t)/2$ and let $\boldsymbol{\omega}_i^d(t) = R_{i1}^d(t)\left(\boldsymbol{\omega}_1^d(t) + \boldsymbol{\omega}_{1i}^d(t)\right)$

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for $i = 2, \dots, n$. Similarly, whenever there is no confusion, we drop the explicit dependence of the desired absolute angular velocity on time.

We consider relative attitude formation control at the kinematic level, which is described as follows.

Relative attitude formation problem: construct an angular velocity controller for each agent to make the relative attitude between any two agents converges to the desired one, that is, $R_{ij} \to R_{ij}^d$ for any $i, j \in \mathcal{V}$ as time tends to infinity.

As a special case, we also consider the *static relative attitude formation* problem, that is, the desired relative attitude R_{ij}^d for any agent pair (i, j) is independent of time and therefore is constant. In this case, we choose $\boldsymbol{\omega}_i^d(\cdot) \equiv \mathbf{0}_3$ on $[0, \infty)$ for any $i \in \mathcal{V}$.

3.2. Error attitudes

For any ordered agent pair (i, j), we define the error attitude between the actual relative attitude R_{ij} and the desired relative attitude R_{ij}^d as $E_{ij} = R_{ij} R_{ji}^d$. Denote $(\theta_{ij}, \boldsymbol{k}_{ij})$ as the axis and angle of the error rotation E_{ij}. Apparently, the desired formation is attained when $E_{ij} = I_3$ or $\theta_{ij} = 0$ for any $i \neq j$.

From the rotation kinematics (12) and (15), the motion equations for the error attitudes are governed by

$$\dot{E}_{ij} = \widehat{\boldsymbol{\omega}}_{ij} E_{ij} - E_{ij} \widehat{\boldsymbol{\omega}}_{ij}^d(t), \quad \forall i, j \in \mathcal{V}, i \neq j. \tag{17}$$

It is worthwhile to notice that the error system (17) is nonautonomous because the desired formation (13) is time varying.

We use $\boldsymbol{E} = \{E_{ij}\}_{i,j \in \mathcal{V}, i \neq j} \in SO(3)^{n(n-1)}$ to denote the state of the system (17), where the product manifold $SO(3)^{n(n-1)}$ is the $n(n-1)$-fold Cartesian product of $SO(3)$ with itself. From (14), the state $\boldsymbol{E}$ should satisfy the following compatible conditions:

$$E_{ij} = R_{ij}^d E_{ji}^T R_{ji}^d = R_{ij} E_{ji}^T R_{ji}, \quad R_{ij}^d E_{jk} R_{ji}^d = E_{ij}^T E_{ik}, \quad \forall i, j, k. \tag{18}$$

Let $\mathcal{D} \subset SO(3)^{n(n-1)}$ be the set of states satisfying (18), which is the state space of the system (17). We use the metric in $\mathcal{D}$ as

$$d_{\mathcal{D}}(\boldsymbol{E}, \bar{\boldsymbol{E}}) = \max_{i,j \in \mathcal{V}} d_{SO(3)}(E_{ij}, \bar{E}_{ij}), \quad \forall \boldsymbol{E}, \bar{\boldsymbol{E}} \in \mathcal{D}.$$

Denote $\boldsymbol{E}^e \in \mathcal{D}$ as the state such that $E_{ij} = I_3$ for any $i \neq j$. Then the desired formation is achieved if and only if $\boldsymbol{E}(t) \to \boldsymbol{E}^e$, or equivalently $d_{\mathcal{D}}(\boldsymbol{E}(t), \boldsymbol{E}^e) \to 0$, as $t \to \infty$.

In the paper, we use a geodesic distance dependent function $h : [0, \pi] \to \mathbb{R}$ satisfying the following assumption.

Assumption 3.1
$h(\cdot)$ is twice continuously differentiable on $[0, \pi]$, $h(0) = h'(0) = 0$ and $h'(\cdot) > 0$ on $(0, \pi)$.

Remark 3.2
Here are some examples of $h(\cdot)$ satisfying Assumption 3.1:

(i) $h(\theta) = \theta^2/2$, $h'(\theta) = \theta$;
(ii) $h(\theta) = 2\sin^2(\theta/2)$, $h'(\theta) = \sin\theta$;
(iii) $h(\theta) = 4\sin^2(\theta/4)$, $h'(\theta) = \sin(\theta/2)$;
(iv) $h(\theta) = -4\log(\cos(\theta/4))$, $h'(\theta) = \tan(\theta/4)$.

For a rotation $R \in SO(3)$ with $(\theta, \boldsymbol{k})$ as its angle and axis, $h'(\theta)\boldsymbol{k}$ actually is a three-parameter representation of the rotation. Examples (i) and (iv) correspond to the rotation vector representation [5] and modified Rodrigues parameters [21], respectively. And examples (ii) and (iii) are discussed in [26].

The following lemma gives some useful properties for the error attitudes in the state space. The proof is given in Appendix A.

Lemma 3.3
The following statements are satisfied for the error attitude $\boldsymbol{E} \in \mathcal{D}$.

(i) For any $i, j \in \mathcal{V}$, $\theta_{ij} = \theta_{ji}$. If $\theta_{ij} < \pi$, then

$$h'(\theta_{ij})\boldsymbol{k}_{ij} = -h'(\theta_{ji})R_{ij}^d\boldsymbol{k}_{ji} = -h'(\theta_{ji})R_{ij}\boldsymbol{k}_{ji}. \tag{19}$$

(ii) For any $i, j, k \in \mathcal{V}$,

$$\theta_{jk} \leqslant \theta_{ij} + \theta_{ik} \tag{20}$$

always holds. Suppose without loss of generality that $\theta_{ij} \geqslant \theta_{ik}$ and $\theta_{ij} \geqslant \theta_{jk}$. Then if $\theta_{ij} + \theta_{kj} + \theta_{ik} < 2\pi$,

$$h'(\theta_{ij})h'(\theta_{ik})\boldsymbol{k}_{ij}^T\boldsymbol{k}_{ik} \geqslant 0, \qquad h'(\theta_{ji})h'(\theta_{jk})\boldsymbol{k}_{ji}^T\boldsymbol{k}_{jk} \geqslant 0, \tag{21}$$

where the equal sign in the first and second inequality holds if and only if $E_{ik} = I_3$ and $E_{jk} = I_3$, respectively.

Remark 3.4
The inequalities in (21) result from the geometry of $SO(3)$. Examples that the inner products in (21) have a negative sign when $\theta_{ij} + \theta_{kj} + \theta_{ik} \geqslant 2\pi$ are given as follows:

(i) $E_{ij} = \exp(-\alpha_1\widehat{\boldsymbol{u}})$ and $E_{ik} = \exp(\alpha_2\widehat{\boldsymbol{u}})$, where $\boldsymbol{u} \in \mathcal{S}^2$, $\alpha_1, \alpha_2 \in (0, \pi)$ and $\alpha_1 + \alpha_2 > \pi$. Then from (18), $\theta_{jk} = 2\pi - \alpha_1 - \alpha_2$ and $R_{ij}^d\boldsymbol{k}_{jk} = -\boldsymbol{u}$. In this case, $\theta_{ij} + \theta_{kj} + \theta_{ik} = 2\pi$ and

$$h'(\theta_{ij})h'(\theta_{ik})\boldsymbol{k}_{ij}^T\boldsymbol{k}_{ik} = -h'(\alpha_1)h'(\alpha_2) < 0,$$
$$h'(\theta_{ji})h'(\theta_{jk})\boldsymbol{k}_{ji}^T\boldsymbol{k}_{jk} = -h'(\alpha_1)h'(2\pi - \alpha_1 - \alpha_2) < 0.$$

(ii) $E_{ij} = \exp(\alpha\widehat{\boldsymbol{u}}_1)$ and $E_{ik} = \exp(\alpha\widehat{\boldsymbol{u}}_2)$, where $\alpha \in (2\pi/3, \pi)$, $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathcal{S}^2$ and $\boldsymbol{u}_1^T\boldsymbol{u}_2 = -\cos(\alpha/2)/(1 - \cos(\alpha/2))$. Then from (18) and Lemma 2.3, $\theta_{jk} = \alpha$ and $R_{ij}^d\boldsymbol{k}_{jk} = \cos(\alpha/2)(\boldsymbol{u}_1 - \boldsymbol{u}_2) + \sin(\alpha/2)\widehat{\boldsymbol{u}}_1\boldsymbol{u}_2$. In this case, $\theta_{ij} + \theta_{kj} + \theta_{ik} = 3\alpha > 2\pi$ and

$$h'(\theta_{ij})h'(\theta_{ik})\boldsymbol{k}_{ij}^T\boldsymbol{k}_{ik} = h'(\alpha)^2\boldsymbol{u}_1^T\boldsymbol{u}_2 < 0,$$
$$h'(\theta_{ji})h'(\theta_{jk})\boldsymbol{k}_{ji}^T\boldsymbol{k}_{jk} = -h'(\alpha)^2\cos(\alpha/2)(1 - \boldsymbol{u}_1^T\boldsymbol{u}_2) < 0.$$

4. ATTITUDE FORMATION WITH SWITCHING TOPOLOGIES

In this section, we suppose the interaction topology for the multi-agent system is time varying with all possible inter-agent topologies represented by digraphs $\mathcal{G}_k = (\mathcal{V}, \mathcal{E}_k)$, $k = 1, 2, \ldots, m$, where the edge $(j, i) \in \mathcal{E}_k$ indicates that agent i can obtain the relative attitude R_{ij} when the underlying interaction graph is $\mathcal{G}_k$. This makes sense in many cases. Actually, in some practical situation, when agent i takes a vision camera as its sensor to observe the motion of its neighbors, the attitude measurement can be extracted from image processing [18]. Then under this assumption, the error attitude E_{ij} is available to agent i if $j \in \mathcal{N}_i$ at any given time.

Let $\mathcal{G}_{\sigma(t)}$ be the inter-agent topology of the system, where $\sigma(t) : [0, \infty) \to \{1, 2, \ldots, m\}$ is a right continuous piecewise constant switching function. As usual, we assume there is a non-vanishing dwell time, denoted by $\tau_d > 0$ for $\sigma(t)$, as a lower bound between any two consecutive switching times, that is, the switching instances $\{\tau_l \mid l = 1, 2, \ldots\}$ satisfy $\inf_l(\tau_{l+1} - \tau_l) \geqslant \tau_d$. Define the union graph of $\mathcal{G}_{\sigma(t)}$ during a time interval $[t_1, t_2)$ as $\mathcal{G}([t_1, t_2)) = \cup_{t\in[t_1,t_2)}\mathcal{G}_{\sigma(t)}$. The inter-agent graph $\mathcal{G}_{\sigma(t)}$ is said to be JQSC if there is a constant $T > 0$ such that $\mathcal{G}([t, t + T))$ is QSC for any $t \geqslant 0$, which is widely used in [2, 5, 14, 15].

In the following, we will first propose a distributed algorithm for the relative attitude formation problem and then give a rigorous stability analysis for the desired state $\boldsymbol{E}^e$.

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4.1. Control design

Define the local potential function for agent $i \in \mathcal{V}$ at time instance t as

$$\varphi_i(R_i) = \sum_{j \in \mathcal{N}_i(t)} h(\theta_{ij}) = \sum_{j \in \mathcal{N}_i(t)} h(d_{SO(3)}(R_i, R_j R_{ji}^d)).$$

From Lemma 2.2, the gradient of $\varphi_i(R_i)$ with respect to the absolute rotation R_i on $SO(3)$ is given by

$$\nabla_{R_i}\varphi_i = -R_i \sum_{j \in \mathcal{N}_i(t)} h'(\theta_{ij})\widehat{\boldsymbol{k}}_{ij}.$$

Notice that if $h'(\pi) \neq 0$, the gradient $\nabla_{R_i}\varphi_i$ at time t actually does not exist when $\theta_{ij} = \pi$ for some $j \in \mathcal{N}_i(t)$. Then combining the kinematics (12) with a basic idea of descent gradient algorithm, we propose the following neighbor-based feedback control law:

$$\boldsymbol{\omega}_i = \boldsymbol{\omega}_i^d(t) + \sum_{j \in \mathcal{N}_i(t)} h'(\theta_{ij})\boldsymbol{k}_{ij}, \quad i = 1, 2, \ldots, n, \tag{22}$$

where the term $\boldsymbol{\omega}_i^d$ is used to compensate the impact caused by the dynamics (15) of the desired relative attitude formation. When $\theta_{ij} = \pi$ for some $j \in \mathcal{N}_i(t)$, from (1) and (9), there are two opposite axes $\boldsymbol{k}_{ij}$ for the rotation E_{ij}, and they satisfy $E_{ij} = -I_3 + 2\boldsymbol{k}_{ij}\boldsymbol{k}_{ij}^T$. Hence, $\boldsymbol{k}_{ij}$ can be derived by computing the unit normal vector of the plane formed by the null space of $E_{ij} + I_3$, and here, we stipulate that the axis $\boldsymbol{k}_{ij}$ in the control (22) is chosen as the one in the upper-left side of the plane. The control law (22) is thus uniquely defined for every possible rotation of the system.

4.2. Stability analysis

The desired state $\boldsymbol{E}^e$ is an equilibrium of the closed-loop system composed of motion (17) and controllers (22). In this section, we analyze the asymptotic stability of this desired equilibrium.

For any unordered agent pair (i, j), we define $V_{ij}(\boldsymbol{E})$ as

$$V_{ij}(\boldsymbol{E}) = h(\theta_{ij}).$$

Because of the angular velocity controllers (22) are piecewise continuous in time, $V_{ij}(\boldsymbol{E})$ along the trajectory of the closed-loop system (17) with (22) is continuous and piecewise differentiable in time. We therefore consider the upper-right Dini time derivative of $V_{ij}(\boldsymbol{E})$ along the trajectory of the closed-loop system (details about Dini derivative can be found in [27, 28]). Suppose $\theta_{ij} < \pi$, then from (2), (6), (10), (11), (16), (17), (19), Lemma 2.2, and the identity $E_{ij}\boldsymbol{k}_{ij} = \boldsymbol{k}_{ij}$, we obtain

$$\begin{aligned}
D^+V_{ij}(\boldsymbol{E}) &= \langle \nabla_{E_{ij}} V_{ij}, D^+E_{ij} \rangle_{E_{ij}} \\
&= \langle E_{ij} h'(\theta_{ij})\widehat{\boldsymbol{k}}_{ij}, \widehat{\boldsymbol{\omega}}_{ij} E_{ij} - E_{ij}\widehat{\boldsymbol{\omega}}_{ij}^d \rangle_{E_{ij}} \\
&= \frac{1}{2}\mathrm{tr}\left[-h'(\theta_{ij})\widehat{\boldsymbol{k}}_{ij} E_{ij}^T \left(\widehat{\boldsymbol{\omega}}_{ij} E_{ij} - E_{ij}\widehat{\boldsymbol{\omega}}_{ij}^d\right)\right] \\
&= h'(\theta_{ij})\boldsymbol{k}_{ij}^T \left(R_{ij}\boldsymbol{\omega}_j - \boldsymbol{\omega}_i - R_{ij}^d\boldsymbol{\omega}_j^d + \boldsymbol{\omega}_i^d\right) \\
&= -h'(\theta_{ij})\boldsymbol{k}_{ij}^T \sum_{k \in \mathcal{N}_i(t)} h'(\theta_{ik})\boldsymbol{k}_{ik} - h'(\theta_{ji})\boldsymbol{k}_{ji}^T \sum_{k \in \mathcal{N}_j(t)} h'(\theta_{jk})\boldsymbol{k}_{jk}.
\end{aligned} \tag{23}$$

Let

$$V(\boldsymbol{E}) = \max_{(i,j)} V_{ij}(\boldsymbol{E}) = h(d_{\mathcal{D}}(\boldsymbol{E}, \boldsymbol{E}^e)),$$

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and $\mathcal{I}(\boldsymbol{E}) = \{(i,j) \,|\, V(\boldsymbol{E}) = V_{ij}(\boldsymbol{E})\}$. Because $h(\cdot)$ is strictly increasing by Assumption 3.1, $\mathcal{I}(\boldsymbol{E})$ is the set of agent pairs possessing the maximal geodesic distance between the actual relative attitudes and the desired ones. And we have that $D^+V(\boldsymbol{E}) = \max_{(i,j)\in\mathcal{I}(\boldsymbol{E})} D^+V_{ij}(\boldsymbol{E})$. Denote

$$\mathcal{R} = \{\boldsymbol{E} \in \mathcal{D} \,|\, d_{\mathcal{D}}(\boldsymbol{E}, \boldsymbol{E}^e) < 2\pi/3\}.$$

In the following, we will first analyze $D^+V(\boldsymbol{E})$ for any $\boldsymbol{E} \in \mathcal{R}$ in Lemma 4.1, which follows directly from Lemma 3.3. Then we discuss the solutions of (17) with (22) starting from $\mathcal{R}$ at any time in Lemma 4.2.

Lemma 4.1
For any $\boldsymbol{E} \in \mathcal{R}$ and $t \geqslant 0$, $D^+V_{ij}(\boldsymbol{E}) \leqslant 0$ for any $(i,j) \in \mathcal{I}(\boldsymbol{E})$ and $D^+V(\boldsymbol{E}) \leqslant 0$. Moreover, for any $(i,j) \in \mathcal{I}(\boldsymbol{E})$, $D^+V_{ij}(\boldsymbol{E}) = 0$ at time t if and only if $E_{ik}(t) = I_3$ for any $k \in \mathcal{N}_i(t)$ and $E_{jk}(t) = I_3$ for any $k \in \mathcal{N}_j(t)$.

Lemma 4.2
For any $\bar{\boldsymbol{E}} \in \mathcal{R}$ and $t_0 \geqslant 0$, the trajectory $\boldsymbol{E}(t)$ of the closed-loop system composed of (17) and (22) starting from $\bar{\boldsymbol{E}}$ at t_0 is defined for all $t \geqslant t_0$ and is unique; $\boldsymbol{E}(t) \in \mathcal{R}$ for any $t \geqslant t_0$ and $V(\boldsymbol{E}(\cdot))$ is non-increasing on $[t_0, \infty)$. Furthermore, as a function of t_0 and $\bar{\boldsymbol{E}}$, the trajectory is continuous in t_0 and $\bar{\boldsymbol{E}}$ on $[t_0, \infty) \times \mathcal{R}$.

Proof
Because $\bar{\boldsymbol{E}} \in \mathcal{R}$, $D^+V(\boldsymbol{E}(t_0)) \leqslant 0$ by Lemma 4.1. Then it holds that $\boldsymbol{E}(t) \in \mathcal{R}$ for any $t \geqslant t_0$ and $V(\boldsymbol{E}(\cdot))$ is non-increasing on $[t_0, \infty)$. Because of the right-hand side of the closed-loop system (17) with (22) is piecewise continuous in t and Lipschitz in $\boldsymbol{E}$ on $[t_0, \infty) \times \mathcal{R}$, the conclusion follows consequently (see [29], Theorems 3.3 and 3.5). □

Then take any $t_1 \geqslant 0$ and suppose $\boldsymbol{E}(t_1) \in \mathcal{R}$. Lemma 4.2 shows that $\boldsymbol{E}(t) \in \mathcal{R}$ for any $t \geqslant t_1$ and $V(\boldsymbol{E}(\cdot))$ is non-increasing on $[t_1, \infty)$. Let $\alpha = V(\boldsymbol{E}(t_1))$ and define another set of agent pairs as

$$\mathcal{K}_\alpha(t) = \{(i,j) \,|\, V_{ij}(\boldsymbol{E}(t)) = \alpha\}, \quad t \geqslant t_1.$$

Apparently, $\mathcal{K}_\alpha(t_1) \neq \emptyset$. Then we give the following three lemmas. In fact, Lemma 4.3 shows that any agent pair not in $\mathcal{K}_\alpha$ at t_1 will never enter it, and once any agent pair in $\mathcal{K}_\alpha$ leaves the set, the pair will never return. Following that, Lemma 4.4 displays that if $\boldsymbol{E}(t_1) \neq \boldsymbol{E}^e$, then $\mathcal{K}_\alpha$ is empty at any $t \geqslant t_1 + (n-1)T$ under certain mild conditions on the connectedness of the inter-agent graph $\mathcal{G}_{\sigma(t)}$, while Lemma 4.5 further shows that $V(\boldsymbol{E}(\cdot))$ on the time interval $[t_1 + (n-1)T + \tau_d, \infty)$ has an upper bound strictly less than α and independent of t_1. The proofs are given in Appendix B.

Lemma 4.3
For any agent pair (i,j), if there is a moment $t' \geqslant t_1$ such that $(i,j) \notin \mathcal{K}_\alpha(t')$, then $(i,j) \notin \mathcal{K}_\alpha(t)$ for any $t \geqslant t'$.

Lemma 4.4
Suppose $\mathcal{G}_{\sigma(t)}$ is JQSC with time constant T, that is, $\mathcal{G}([t, t+T))$ is QSC for any $t \geqslant 0$. If $\boldsymbol{E}(t_1) \in \mathcal{R}\backslash\{\boldsymbol{E}^e\}$, then $\mathcal{K}_\alpha(t) = \emptyset$ for any $t \geqslant t_1 + (n-1)T$.

Lemma 4.5
Suppose $\mathcal{G}_{\sigma(t)}$ is JQSC with time constant T and dwell time $\tau_d > 0$. If $\boldsymbol{E}(t_1) \in \mathcal{R}\backslash\{\boldsymbol{E}^e\}$ and $V(\boldsymbol{E}(t_1)) = \alpha$, then there exists a $\delta_\alpha > 0$, independent of t_1, such that $V(\boldsymbol{E}(t)) \leqslant \alpha - \delta_\alpha$ for any $t \geqslant t_1 + (n-1)T + \tau_d$.

We are now ready to give the main result of this section, that is, when the initial relative attitude $R_{ij}(t_0)$ is contained within the open geodesic ball $B_{2\pi/3}(R_{ij}^d(t_0))$ for any agent pair (i,j), the desired relative attitude formation is achieved eventually provided that the inter-agent graph is JQSC. In other words, as $t \to \infty$, any agent $i \in \mathcal{V}$ will rotate itself with the desired absolute angular velocity $\boldsymbol{\omega}_i^d(t)$ and at the same time maintain the relative attitudes to other agents as the desired ones.

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Theorem 4.6
Suppose $\mathcal{G}_{\sigma(t)}$ is JQSC. The desired equilibrium $\boldsymbol{E}^e$ is uniformly asymptotically stable with respect to the closed-loop system composed of (17) and (22). Moreover, $\mathcal{R}$ is contained in the region of attraction.

Proof
We first show that $\boldsymbol{E}^e$ is uniformly stable. For any $\varepsilon \in (0, \pi]$ and any $t_0 \geqslant 0$, let $\delta = \min\{\varepsilon, 2\pi/3\}$. Take any initial state $\boldsymbol{E}(t_0) \in \mathcal{D}$ satisfying $d_{\mathcal{D}}(\boldsymbol{E}(t_0), \boldsymbol{E}^e) < \delta$. Because $\delta < 2\pi/3$, $V(\boldsymbol{E}(\cdot))$ is non-increasing on $[t_0, \infty)$ by Lemma 4.2. Hence, $d_{\mathcal{D}}(\boldsymbol{E}(t), \boldsymbol{E}^e) \leqslant d_{\mathcal{D}}(\boldsymbol{E}(t_0), \boldsymbol{E}^e) < \delta \leqslant \varepsilon$ for any $t \geqslant t_0$ by Assumption 3.1. This implies $\boldsymbol{E}^e$ is uniformly stable.

In the following, we use $\boldsymbol{E}(t; t_0, \bar{\boldsymbol{E}})$ to denote the trajectory of the closed-loop system starting from $\bar{\boldsymbol{E}} \in \mathcal{D}$ at time $t_0 \geqslant 0$. Take any $\bar{\boldsymbol{E}} \in \mathcal{R}$ and any $t_0 \geqslant 0$, we next show that $\lim_{t\to\infty} \boldsymbol{E}(t; t_0, \bar{\boldsymbol{E}}) = \boldsymbol{E}^e$.

From Lemma 4.2, $\boldsymbol{E}(t; t_0, \bar{\boldsymbol{E}}) \in \mathcal{R}$ for any $t \geqslant t_0$ and $V(\boldsymbol{E}(\cdot; t_0, \bar{\boldsymbol{E}}))$ is non-increasing on $[t_0, \infty)$. Then because of $V(\boldsymbol{E}(\cdot; t_0, \bar{\boldsymbol{E}})) \geqslant 0$ on $[t_0, \infty)$ by Assumption 3.1, there is a constant $\alpha \geqslant 0$ such that $\lim_{t\to\infty} V(\boldsymbol{E}(t; t_0, \bar{\boldsymbol{E}})) = \alpha$. Because the trajectory $\boldsymbol{E}(t; t_0, \bar{\boldsymbol{E}}) \subset \mathcal{R}$ is bounded, its positive limit set denoted by $L(t_0, \bar{\boldsymbol{E}}) \subset \mathcal{R}$ is nonempty. For any limit point $\tilde{\boldsymbol{E}} \in L(t_0, \bar{\boldsymbol{E}})$, there exists an increasing time sequence $\{t_k\}$ such that as $k \to \infty$, $t_k \to \infty$, and $\boldsymbol{E}(t_k; t_0, \bar{\boldsymbol{E}}) \to \tilde{\boldsymbol{E}}$. Then we have that $V(\tilde{\boldsymbol{E}}) = \alpha$ by the continuity of V.

Take any $\tilde{\boldsymbol{E}} \in L(t_0, \bar{\boldsymbol{E}})$, and suppose $\tilde{\boldsymbol{E}} \neq \boldsymbol{E}^e$. Because $V(\tilde{\boldsymbol{E}}) = \alpha$ and $\tilde{\boldsymbol{E}} \in \mathcal{R}\backslash\{\boldsymbol{E}^e\}$, from Lemma 4.5, there is a $\delta_\alpha > 0$ such that

$$V(\boldsymbol{E}(\bar{t}_0 + (n-1)T + \tau_d; \bar{t}_0, \tilde{\boldsymbol{E}})) \leqslant \alpha - \delta_\alpha, \quad \forall \bar{t}_0 \in [t_0, \infty). \tag{24}$$

Because $\tilde{\boldsymbol{E}} \in L(t_0, \bar{\boldsymbol{E}})$, for any $\varepsilon > 0$, there is a moment $t_1 \geqslant t_0$ such that $d_{\mathcal{D}}(\boldsymbol{E}(t_1; t_0, \bar{\boldsymbol{E}}), \tilde{\boldsymbol{E}}) < \varepsilon$. On the other hand, from Lemma 4.2, the trajectory of the closed-loop system starting from $\mathcal{R}$ is continuous in the initial state. Hence, for the positive number δ_α in (24), there exist $\varepsilon > 0$ and $t_1 \geqslant t_0$ such that $d_{\mathcal{D}}(\boldsymbol{E}(t_1; t_0, \bar{\boldsymbol{E}}), \boldsymbol{E}(t_1; t_1, \tilde{\boldsymbol{E}})) < \varepsilon$, and

$$d_{\mathcal{D}}(\boldsymbol{E}(t_2; t_0, \bar{\boldsymbol{E}}), \boldsymbol{E}(t_2; t_1, \tilde{\boldsymbol{E}})) < \delta_\alpha/(2\eta), \tag{25}$$

where $t_2 = t_1 + (n-1)T + \tau_d$ and $\eta = \max_{\theta\in[0,2\pi/3]} h'(\theta)$. Notice that η exists and is positive by Assumption 3.1.

For any $\boldsymbol{E}^1, \boldsymbol{E}^2 \in \mathcal{R}$, we have that

$$\begin{aligned} V(\boldsymbol{E}^1) - V(\boldsymbol{E}^2) &= h(d_{\mathcal{D}}(\boldsymbol{E}^1, \boldsymbol{E}^e)) - h(d_{\mathcal{D}}(\boldsymbol{E}^2, \boldsymbol{E}^e)) \\ &\leqslant \eta \left| d_{\mathcal{D}}(\boldsymbol{E}^1, \boldsymbol{E}^e) - d_{\mathcal{D}}(\boldsymbol{E}^2, \boldsymbol{E}^e) \right| \leqslant \eta\, d_{\mathcal{D}}(\boldsymbol{E}^1, \boldsymbol{E}^2). \end{aligned}$$

Then from (25), $V(\boldsymbol{E}(t_2; t_0, \bar{\boldsymbol{E}})) - V(\boldsymbol{E}(t_2; t_1, \tilde{\boldsymbol{E}})) < \delta_\alpha/2$. Because $V(\boldsymbol{E}(t_2; t_1, \tilde{\boldsymbol{E}})) \leqslant \alpha - \delta_\alpha$ by (24), this implies $V(\boldsymbol{E}(t_2; t_0, \bar{\boldsymbol{E}})) < \alpha - \delta_\alpha/2$, which contradicts the fact that α is the lower bound of $V(\boldsymbol{E}(\cdot; t_0, \bar{\boldsymbol{E}}))$ on $[t_0, \infty)$. Hence, $L(t_0, \bar{\boldsymbol{E}}) = \{\boldsymbol{E}^e\}$, which implies $\lim_{t\to\infty} \boldsymbol{E}(t; t_0, \bar{\boldsymbol{E}}) = \boldsymbol{E}^e$. The proof is completed. □

Theorem 4.6 shows that the desired equilibrium $\boldsymbol{E}^e$ is asymptotically stable with the region of attraction containing $\mathcal{R}$ provided that the inter-agent topology is JQSC. Because of the geometry of $SO(3)$ and the uncertainty of the topology graph, the region of attraction for the desired equilibrium $\boldsymbol{E}^e$ in Theorem 4.6 is the best we can estimate for the general formation problem.

5. ATTITUDE FORMATION WITH FIXED TOPOLOGY

To obtain better results about the convergence rate and the convergence region, we discuss some special cases of the relative attitude formation problem. In this section, we suppose the inter-agent graph is fixed and denote it simply by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. We give the sufficient and necessary condition for the desired relative attitude formation to be locally exponentially convergent in Section 5.1, and give several sufficient conditions for the static relative attitude formation to be achieved from almost all initial rotations in Section 5.2.

5.1. Exponential convergence

When the inter-agent graph is fixed, the control law (22) can be written as

$$\boldsymbol{\omega}_i = \boldsymbol{\omega}_i^d(t) + \sum_{j\in\mathcal{N}_i} h'(\theta_{ij})\boldsymbol{k}_{ij}, \quad i = 1, 2, \ldots, n. \tag{26}$$

From Theorem 4.6, the desired equilibrium $\boldsymbol{E}^e$ is uniformly asymptotically stable with respect to the closed-loop system composed of (17) and (26) provided that $\mathcal{G}$ is QSC. We then demonstrate that $\boldsymbol{E}^e$ is further exponentially stable when $h''(0) > 0$. Notice that the exponential stability of $\boldsymbol{E}^e$ is robust with respect to a certain class of perturbations (see [29], Lemma 9.1).

Theorem 5.1

The desired equilibrium $\boldsymbol{E}^e$ of the closed-loop system given by motion equations (17) with controllers (26) is exponentially stable if and only if $\mathcal{G}$ is QSC and $h''(0) > 0$.

Proof

We linearize the closed-loop equations about $\boldsymbol{E}^e$ (see [30] for the linearization of dynamics on $SO(3)$) and show that the corresponding equilibrium of the linearization system is exponentially stable if and only if $\mathcal{G}$ is QSC and $h''(0) > 0$.

Let $\varepsilon \in \mathbb{R}$ be a perturbation parameter, and suppose the initial state of the system is a perturbation of $\boldsymbol{E}^e$ denoted by $\boldsymbol{E}^\varepsilon = \{\exp(\varepsilon\alpha_{ij}\widehat{\boldsymbol{u}}_{ij})\}_{i,j\in\mathcal{V},i\neq j} \in \mathcal{D}$, where $\alpha_{ij} > 0$ and $\boldsymbol{u}_{ij} \in \mathcal{S}^2$. Let $\boldsymbol{E}(t;\varepsilon)$ be the trajectory of the closed-loop system starting form $\boldsymbol{E}^\varepsilon$ at $t = 0$. Then we denote $\boldsymbol{\omega}_i(t;\varepsilon)$ as the angular velocity of agent i and $\theta_{ij}(t;\varepsilon)$ as the angle of the error attitude $E_{ij}(t;\varepsilon)$. Notice that when $\varepsilon = 0$, $\boldsymbol{E}^\varepsilon = \boldsymbol{E}^e$. Hence, $\boldsymbol{E}(t;0) = \boldsymbol{E}^e$ and $\boldsymbol{\omega}_i(t;0) = \boldsymbol{\omega}_i^d(t)$ for any $t \geqslant 0$.

Let $E_{ij\varepsilon}(t) = \left(\partial E_{ij}(t;\varepsilon)/\partial\varepsilon\right)\big|_{\varepsilon=0}$. Then $E_{ij\varepsilon} \in so(3)$ because of $E_{ij\varepsilon} \in T_{E_{ij}(t;0)}SO(3)$. We denote $\boldsymbol{x}_{ij} = \left(E_{ij\varepsilon}\right)^\vee$, and take $\{\boldsymbol{x}_{ij}\}_{i,j\in\mathcal{V},i\neq j}$ as the state of the linearization system.

From the control law (26) and the identity (8), the angular velocity $\boldsymbol{\omega}_i(t;\varepsilon)$ satisfies

$$\boldsymbol{\omega}_i(t;\varepsilon) = \boldsymbol{\omega}_i^d(t) + \sum_{k\in\mathcal{N}_i} \frac{h'(\theta_{ik}(t;\varepsilon))}{2\sin(\theta_{ik}(t;\varepsilon))}\left[E_{ik}(t;\varepsilon) - E_{ik}^T(t;\varepsilon)\right]^\vee.$$

Let $\boldsymbol{\omega}_{i\varepsilon}(t) = (\partial\boldsymbol{\omega}_i(t;\varepsilon))/\partial\varepsilon)|_{\varepsilon=0}$. Then from the previous equation, we obtain

$$\boldsymbol{\omega}_{i\varepsilon} = h''(0)\sum_{k\in\mathcal{N}_i}\boldsymbol{x}_{ik}. \tag{27}$$

Because the trajectory $E_{ij}(t;\varepsilon)$ satisfies the following differential equation

$$\dot{E}_{ij}(t;\varepsilon) = \left[E_{ij}(t;\varepsilon)R_{ij}^d(t)\boldsymbol{\omega}_j(t;\varepsilon) - \boldsymbol{\omega}_i(t;\varepsilon) - E_{ij}(t;\varepsilon)\boldsymbol{\omega}_{ij}^d(t)\right]^\wedge E_{ij}(t;\varepsilon),$$

it follows that $\dot{E}_{ij\varepsilon} = \left(R_{ij}^d(t)\boldsymbol{\omega}_{j\varepsilon} - \boldsymbol{\omega}_{i\varepsilon} + E_{ij\varepsilon}\boldsymbol{\omega}_i^d(t)\right)^\wedge$. Then substituting (27), we obtain the linearization system as follows:

$$\dot{\boldsymbol{x}}_{ij} = h''(0)R_{ij}^d(t)\sum_{k\in\mathcal{N}_j}\boldsymbol{x}_{jk} - h''(0)\sum_{k\in\mathcal{N}_i}\boldsymbol{x}_{ik} - \widehat{\boldsymbol{\omega}}_i^d(t)\boldsymbol{x}_{ij}, \quad \forall i,j\in\mathcal{V}, i\neq j. \tag{28}$$

Furthermore, from (18), the perturbed solution satisfies the following compatible conditions:

$$E_{ij}(t;\varepsilon) = R_{ij}^d(t)E_{ji}^T(t;\varepsilon)R_{ji}^d(t), \quad R_{ij}^d(t)E_{jk}(t;\varepsilon)R_{ji}^d(t) = E_{ij}^T(t;\varepsilon)E_{ik}(t;\varepsilon), \quad \forall i,j,k.$$

Differentiating both sides of the aforementioned compatible identities with respect to ε and taking $\varepsilon = 0$ yields

$$\boldsymbol{x}_{ij} = -R_{ij}^d(t)\boldsymbol{x}_{ji}, \quad R_{ij}^d(t)\boldsymbol{x}_{jk} = \boldsymbol{x}_{ik} - \boldsymbol{x}_{ij}, \quad \forall i,j,k,$$

where we stipulate that $\boldsymbol{x}_{ik} = \boldsymbol{0}$ if $k = i$. Thus, $\boldsymbol{x} = [\boldsymbol{x}_{12}^T, \boldsymbol{x}_{13}^T, \ldots, \boldsymbol{x}_{1n}^T]^T \in \mathbb{R}^{3(n-1)}$ is sufficient to represent the state of the linearization system (28), and its motion equation is governed by

$$\dot{\boldsymbol{x}}_{1j} = h''(0) \sum_{k \in \mathcal{N}_j} \left(\boldsymbol{x}_{1k} - \boldsymbol{x}_{1j}\right) - h''(0) \sum_{k \in \mathcal{N}_1} \boldsymbol{x}_{1k} - \widehat{\boldsymbol{\omega}}_1^d(t)\boldsymbol{x}_{1j}, \quad j = 2, 3, \ldots, n. \tag{29}$$

Let $L = [l_{ij}]_{n \times n}$ be the Laplacian of $\mathcal{G}$ and partition L as

$$L = \begin{bmatrix} l_{11} & \boldsymbol{a}^T \\ \boldsymbol{b} & L_{n-1} \end{bmatrix}.$$

Then the linearization system (29) can be written in matrix form as

$$\dot{\boldsymbol{x}} = \left[h''(0)K \otimes I_3 - I_{n-1} \otimes \widehat{\boldsymbol{\omega}}_1^d(t)\right]\boldsymbol{x}, \tag{30}$$

where $K = \mathbf{1}_{n-1}\boldsymbol{a}^T - L_{n-1}$. Because $L\mathbf{1}_n = \mathbf{0}$, we obtain

$$\begin{bmatrix} 1 & \mathbf{0}_{n-1}^T \\ \mathbf{1}_{n-1} & I_{n-1} \end{bmatrix}^{-1} \begin{bmatrix} l_{11} & \boldsymbol{a}^T \\ \boldsymbol{b} & L_{n-1} \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0}_{n-1}^T \\ \mathbf{1}_{n-1} & I_{n-1} \end{bmatrix} = \begin{bmatrix} 0 & \boldsymbol{a}^T \\ \mathbf{0}_{n-1} & -K \end{bmatrix}.$$

Then K is Hurwitz if and only if $\mathcal{G}$ is QSC. Hence, there exists a positive definite symmetric matrix $P \in \mathbb{R}^{(n-1)\times(n-1)}$ such that $K^T P + PK = -I_{n-1}$ (see [29], Theorem 4.6). Notice that because of $P \otimes \widehat{\boldsymbol{\omega}}_1^d$ is skew symmetric, $\boldsymbol{x}^T(P \otimes \widehat{\boldsymbol{\omega}}_1^d)\boldsymbol{x} = 0$ for any $\boldsymbol{x} \in \mathbb{R}^{3(n-1)}$. Denote λ_M and λ_m as the respective largest and smallest eigenvalue of P. Then take the Lyapunov candidate as $V_l(\boldsymbol{x}) = \boldsymbol{x}^T(P \otimes I_3)\boldsymbol{x}$, and compute the time derivative of $V_l(\boldsymbol{x})$ along the linearization system (30), which leads to

$$\dot{V}_l(\boldsymbol{x}) = h''(0)\boldsymbol{x}^T\left[\left(K^T P + PK\right) \otimes I_3\right]\boldsymbol{x} = -h''(0)\boldsymbol{x}^T\boldsymbol{x} \leqslant -V_l(\boldsymbol{x})h''(0)/\lambda_M.$$

Thus, $\|\boldsymbol{x}(t)\| \leqslant (\lambda_M/\lambda_m)\|\boldsymbol{x}(t_0)\|e^{-(t-t_0)h''(0)/\lambda_M}$, and our conclusion follows consequently. □

5.2. *Static relative attitude formation*

In this section, we consider the static relative attitude formation problem, that is, the motion equations (12) of the absolute attitudes with controllers (22) are governed by

$$\dot{R}_i = R_i \sum_{j \in \mathcal{N}_i} h'(\theta_{ij})\widehat{\boldsymbol{k}}_{ij}, \quad i = 1, 2, \ldots, n. \tag{31}$$

Let $\boldsymbol{R} = \{R_i\}_{i \in \mathcal{V}} \in SO(3)^n$ be the state of the closed-loop system (31) and denote the corresponding error attitude by $\boldsymbol{E}(\boldsymbol{R}) = \{E_{ij}(\boldsymbol{R})\}_{i,j \in \mathcal{V}, i \neq j}$. We denote $\mathcal{M}$ as the entire equilibrium set and $\mathcal{M}^e$ as the desired equilibrium set of (31), that is, $\mathcal{M} = \{\boldsymbol{R} \in SO(3)^n \mid \sum_{j \in \mathcal{N}_i} h'(\theta_{ij})\boldsymbol{k}_{ij} = \mathbf{0}_3,\ \forall i \in \mathcal{V}\}$ and $\mathcal{M}^e = \{\boldsymbol{R} \in SO(3)^n \mid \boldsymbol{E}(\boldsymbol{R}) = \boldsymbol{E}^e\}$.

Remark 5.2
For a general graph $\mathcal{G}$, there are undesired equilibria in $\mathcal{M}$. For example, suppose $\mathcal{G}$ is an undirected ring graph and $R_{ij}^d = I_3$ for any $i \neq j$. Then any rotation $\boldsymbol{R}$ with $R_1, \ldots, R_n$ distributing equidistantly on a closed geodesic of $SO(3)$ is also contained in $\mathcal{M}$. In fact, the desired equilibrium set $\mathcal{M}^e \subset \mathcal{M}$ is irrelevant to the inter-agent graph, and the equilibria in $\mathcal{M}\backslash\mathcal{M}^e$ vary with the inter-agent graph.

Let $\Omega_{\mathcal{G},\pi} = \{\boldsymbol{R} \in SO(3)^n \mid \exists (i,j) \in \mathcal{E} \text{ s.t. } \theta_{ij} = \pi\}$. Notice that if $h'(\pi) \neq 0$, the right-hand side of (31) is not continuous in $\boldsymbol{R}$ on $\Omega_{\mathcal{G},\pi}$. Next, we give two lemmas: one gives the set which the trajectories of the closed-loop system (31) from (almost) all $SO(3)^n$ approach, and the other gives the linearization of (31) about an equilibrium. The proofs are given in Appendix C.

Lemma 5.3
Suppose $\mathcal{G}$ is undirected. As time tends to infinity, (i) if $h'(\pi) = 0$, all trajectories of the closed-loop system (31) approach $\mathcal{M}$; (ii) if $h'(\pi) > 0$, the trajectories of (31) starting from almost all $SO(3)^n$ approach $\mathcal{M}\backslash\Omega_{\mathcal{G},\pi}$.

Lemma 5.4
For any equilibrium $\boldsymbol{R} \in \mathcal{M}$ if $h'(\pi) = 0$ or any $\boldsymbol{R} \in \mathcal{M}\backslash\Omega_{\mathcal{G},\pi}$ if $h'(\pi) \neq 0$, the system matrix of the linearization system of the closed-loop system (31) about $\boldsymbol{R}$ is equal to $J(\boldsymbol{R}) \in \mathbb{R}^{3n\times 3n}$ whose (i,j)th, 3×3 block is given by

$$[J(\boldsymbol{R})]_{i,j} = \begin{cases} -\sum_{k\in\mathcal{N}_i} H(E_{ij}(\boldsymbol{R}))^T, & \text{if } j = i \\ H(E_{ij}(\boldsymbol{R}))R^d_{ij}, & \text{if } j \in \mathcal{N}_i \\ O_3, & \text{otherwise} \end{cases},$$

where

$$H(Q) = \frac{1}{2}\cot\frac{\theta}{2}h'(\theta)I_3 + \left[h''(\theta) - \frac{1}{2}\cot\frac{\theta}{2}h'(\theta)\right]\boldsymbol{k}\boldsymbol{k}^T + \frac{1}{2}h'(\theta)\widehat{\boldsymbol{k}}$$

for a rotation $Q \in SO(3)$ whose axis and angle is $(\theta, \boldsymbol{k})$.

The following theorem shows that when $\mathcal{G}$ is an undirected tree, the desired equilibrium set $\mathcal{M}^e$ is almost globally asymptotically stable.

Theorem 5.5
Suppose $\mathcal{G}$ is an undirected tree and $h(\cdot)$ satisfies either (i) $h'(\pi) > 0$ or (ii) $h'(\pi) = 0$ and $h''(\pi) < 0$. The trajectories of the closed-loop system (31) starting from almost all $SO(3)^n$ approach $\mathcal{M}^e$ as time tends to infinity.

Proof
Take any $\boldsymbol{R} \in \mathcal{M}$, and denote $(\theta_{ij}, \boldsymbol{k}_{ij})$ as the angle and axis of $E_{ij}(\boldsymbol{R})$. We first show that

$$h'(\theta_{ij})\boldsymbol{k}_{ij} = \boldsymbol{0}_3, \quad \forall (i,j) \in \mathcal{E}. \tag{32}$$

Because $\mathcal{G}$ is a tree, it has at least two leaves, and we denote anyone of them as p. Because of p only has one neighbor in $\mathcal{G}$ that we designated as q, $\boldsymbol{R} \in \mathcal{M}$ implies $h'(\theta_{pq})\boldsymbol{k}_{pq} = \boldsymbol{0}_3$. Then let $\mathcal{U} = \mathcal{V}\backslash\{p\}$ and $\mathcal{G}_{\mathcal{U}} = \{\mathcal{U}, \mathcal{E}_{\mathcal{U}}\}$ be the induced subgraph of $\mathcal{G}$ by $\mathcal{U}$. Because $\mathcal{E}_{\mathcal{U}} = \mathcal{E}\backslash\{(p,q)\}$ and $h'(\theta_{qp})\boldsymbol{k}_{qp} = -R^d_{qp}h'(\theta_{pq})\boldsymbol{k}_{pq} = \boldsymbol{0}_3$, $\boldsymbol{R} \in \mathcal{M}$ implies $\sum_{j\in\mathcal{U},(i,j)\in\mathcal{E}_{\mathcal{U}}} h'(\theta_{ij})\boldsymbol{k}_{ij} = \boldsymbol{0}_3$ for any $i \in \mathcal{U}$. Because of the subgraph $\mathcal{G}_{\mathcal{U}}$ is still a tree, we can repeat removing a leaf node of the new induced subgraph until there are two nodes in the remaining graph. Then we obtain (32) by the iteration.

Suppose $h'(\pi) > 0$. Because there are undirected paths in $\mathcal{G}$ from any node to every other node, we further obtain $h'(\theta_{ij})\boldsymbol{k}_{ij} = \boldsymbol{0}_3$ for any $i, j \in \mathcal{V}$ by (18) and (32). Because of $h'(\pi) > 0$, $\theta_{ij} = 0$ for any $i, j \in \mathcal{V}$ by Assumption 3.1. Hence, $\boldsymbol{R} \in \mathcal{M}^e$, and the conclusion follows consequently by Lemma 5.3.

Suppose $h'(\pi) = 0$. From Assumption 3.1, (32) implies $\theta_{ij} = 0$ or π for any $(i,j) \in \mathcal{E}$. Suppose there exists an edge $(p,q) \in \mathcal{E}$ with $\theta_{pq} = \pi$, we then show $\boldsymbol{R}$ is unstable. By removing the edge (p,q) in $\mathcal{G}$, the tree graph is divided into two disjoint trees, and we denote the node set of the two trees as $\mathcal{V}_1$ and $\mathcal{V}_2$, respectively. Because $\theta_{pq} = \pi$, there are two axes corresponding to $E_{pq}(\boldsymbol{R})$ opposite each other. We take any one of them and denote it as $\boldsymbol{k}_{pq}$. Let $\boldsymbol{u} = R_p\boldsymbol{k}_{pq}$ and $\boldsymbol{x} = [\boldsymbol{x}_1^T, \ldots, \boldsymbol{x}_n^T]$ with $\boldsymbol{x}_i = R_i^T\boldsymbol{u}$ for $i \in \mathcal{V}_1$ and $\boldsymbol{x}_i = -R_i^T\boldsymbol{u}$ for $i \in \mathcal{V}_2$. For any $(i,j) \in \mathcal{E}$, $H(E_{ij}(\boldsymbol{R}))E_{ij}(\boldsymbol{R})^T = H(E_{ij}(\boldsymbol{R}))^T$ by (9), and $H(\boldsymbol{E}_{ji}(\boldsymbol{R})) = R_{ji}H(\boldsymbol{E}_{ij}(\boldsymbol{R}))^T R_{ij}$ by (19). Then we use them to simplify the expression $\boldsymbol{x}^T J(\boldsymbol{R})\boldsymbol{x}$ and obtain $\boldsymbol{x}^T J(\boldsymbol{R})\boldsymbol{x} = -4h''(\pi) > 0$. Hence, $J(\boldsymbol{R})$ has at least one positive eigenvalue, implying $\boldsymbol{R}$ is unstable. Then the conclusion follows by Lemma 5.3. □

We then generalize the almost global convergent attitude consensus algorithm in [4, 6] to the static relative attitude formation problem in the following theorem.

Theorem 5.6
Suppose $\mathcal{G}$ is undirected and connected, and the function $h(\cdot)$ is chosen as

$$h(\theta) = ah_0(\theta), \quad h_0(\theta) = \frac{1}{b} - \left(\frac{1}{b} + \theta\right)e^{-b\theta}, \tag{33}$$

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where a is any positive constant and $b > 0$ is a sufficiently large constant. As time tends to infinity, the trajectories of the closed-loop system (31) starting from almost all $SO(3)^n$ approach the desired equilibrium set $\mathcal{M}^e$.

Proof

From Theorem 4.6 and Lemma 5.3, because $h'(\pi) = ab\pi e^{-b\pi} > 0$, we only need to show that any $\boldsymbol{R} \in \mathcal{M}\backslash\Omega_{\mathcal{G},\pi}$ is unstable. Take any $\boldsymbol{R} \in \mathcal{M}\backslash\Omega_{\mathcal{G},\pi}$. In the following, we show that the system matrix $J(\boldsymbol{R})$ of the linearization system has at least one positive eigenvalue, and therefore, $\boldsymbol{R}$ is unstable.

Denote θ_{ij} as the angle of the error rotation $E_{ij}(\boldsymbol{R})$. Let $\alpha = 2\pi/(3(n-1))$ and $\mathcal{G}_{\boldsymbol{R},\alpha} = (\mathcal{V}, \mathcal{E}_{\boldsymbol{R},\alpha})$, where $\mathcal{E}_{\boldsymbol{R},\alpha} = \{(i,j) \in \mathcal{E} : \theta_{ij} < \alpha\}$. We then claim that $\mathcal{G}_{\boldsymbol{R},\alpha}$ is not connected. Suppose this is not true, then there is a path with the number of edges less than or equal to $n-1$ between any two distinct nodes p, q in $\mathcal{G}_{\boldsymbol{R},\alpha}$, implying $\theta_{pq} < (n-1)\alpha = 2\pi/3$ by (20), and therefore, $\boldsymbol{E}(\boldsymbol{R}) \in \mathcal{R}$. Because $\boldsymbol{R}$ is an equilibrium of (31) and $\boldsymbol{R} \notin \mathcal{M}^e$, $\boldsymbol{E}(\boldsymbol{R}) \in \mathcal{R}$ contradicts the fact that $\{\boldsymbol{R} \in SO(3)^n : \boldsymbol{E}(\boldsymbol{R}) \in \mathcal{R}\}$ is contained in the region of attraction of $\mathcal{M}^e$ by Theorem 4.6.

Denote $\mathcal{V}_1 \subset \mathcal{V}$ as the node set of a connected component of $\mathcal{G}_{\boldsymbol{R},\alpha}$. Let $\mathcal{V}_2 = \mathcal{V}\backslash\mathcal{V}_1$ and $\mathcal{E}_{12} = \{(i,j) \in \mathcal{E} : i \in \mathcal{V}_1, j \in \mathcal{V}_2\}$. Then $\mathcal{V}_1, \mathcal{V}_2 \neq \emptyset$ due to $\mathcal{G}_{\boldsymbol{R},\alpha}$ is not connected, and $\mathcal{E}_{12} \neq \emptyset$ due to $\mathcal{G}$ is connected. For any $(i,j) \in \mathcal{E}_{12}$, $\theta_{ij} \geqslant \alpha$ holds because $(i,j) \notin \mathcal{E}_{\boldsymbol{R},\alpha}$.

Take any $\boldsymbol{u} \in \mathcal{S}^2$ satisfying $\boldsymbol{u}^T R_i \boldsymbol{k}_{ij} \neq 0$ for any $(i,j) \in \mathcal{E}_{12}$. Notice that such $\boldsymbol{u}$ exists because $\mathcal{E}_{12}$ is a finite set. Let $\boldsymbol{x} = [\boldsymbol{x}_1^T, \ldots, \boldsymbol{x}_n^T]^T$, where $\boldsymbol{x}_i = R_i^T \boldsymbol{u}$ if $i \in \mathcal{V}_1$ and $\boldsymbol{x}_i = -R_i^T \boldsymbol{u}$ otherwise. Then we have that

$$\boldsymbol{x}^T J(\boldsymbol{R})\boldsymbol{x} = -4 \sum_{(i,j)\in\mathcal{E}_{12}} \left[h''(\theta_{ij})(\boldsymbol{u}^T R_i \boldsymbol{k}_{ij})^2 + \frac{1}{2}\cot\frac{\theta_{ij}}{2} h'(\theta_{ij}) \left(1 - (\boldsymbol{u}^T R_i \boldsymbol{k}_{ij})^2\right) \right].$$

Let $y = \min_{(i,j)\in\mathcal{E}_{12}} (\boldsymbol{u}^T R_i \boldsymbol{k}_{ij})^2$, and take any $b > 1/(y\alpha)$. Because $\theta_{ij} \geqslant \alpha > 0$ for any $(i,j) \in \mathcal{E}_{12}$, $\cot(\theta_{ij}/2)/2 < 1/\theta_{ij}$ holds. Therefore, $\boldsymbol{x}^T J(\boldsymbol{R})\boldsymbol{x} > -4\sum_{(i,j)\in\mathcal{E}_{12}} abe^{-b\theta}(1 - b\theta y) > 0$, implying $J(\boldsymbol{R})$ has at least one positive eigenvalue. □

6. NUMERICAL EXAMPLE

In this section, we present numerical examples to show the desired properties of the proposed formation control for a system of five rigid-body agents.

We consider the static relative attitude formation problem. The desired relative attitude formation is assigned as $R_{1i}^d = \exp(\widehat{\boldsymbol{e}}_3(i-1)2\pi/5)$ for $i = 2, \ldots, 5$ and R_{ij}^d for other distinct i, j calculated by (14), where $\boldsymbol{e}_3 = [0, 0, 1]^T$. We take the following three geodesic distance dependent functions satisfying Assumption 3.1:

$$h_1(\theta) = 2k_1 \sin^2(\theta/2), \quad h_2(\theta) = k_2\theta^2, \quad h_3(\theta) = k_3(1/b - (1/b + \theta)e^{-b\theta}), \tag{34}$$

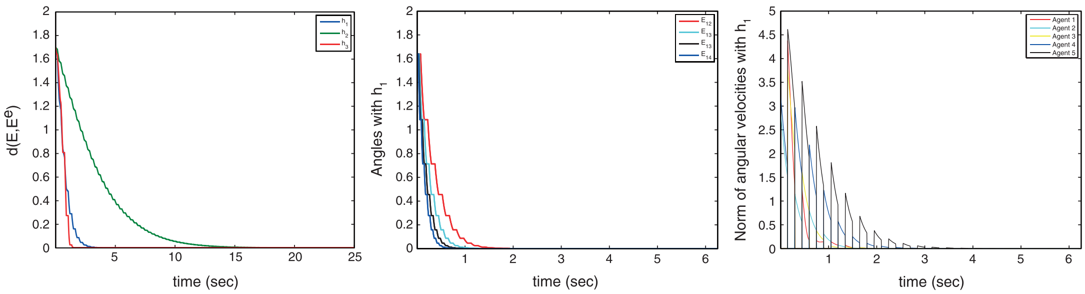
where k_1, k_2, k_3, b are positive constants such that $h_1(\pi) = h_2(\pi) = h_3(\pi)$. It can be verified that the assumptions on the functions in Theorems 4.6, 5.1, 5.5, and 5.6 are all satisfied.

Suppose that there are two inter-agent graphs $\mathcal{G}_i = (\mathcal{V}, \mathcal{E}_i), i = 1, 2$, where the edge set $\mathcal{E}_1 = \{(1,2), (3,4)\}$ and $\mathcal{E}_2 = \{(2,3), (3,1), (4,5)\}$. And suppose the switching signal is given as follows

$$\sigma(t) = \begin{cases} 1, & \text{if } t \in [T_\sigma l, T_\sigma l + T_\sigma/2) \\ 2, & \text{if } t \in [T_\sigma l + T_\sigma/2, (l+1)T_\sigma), \end{cases}$$

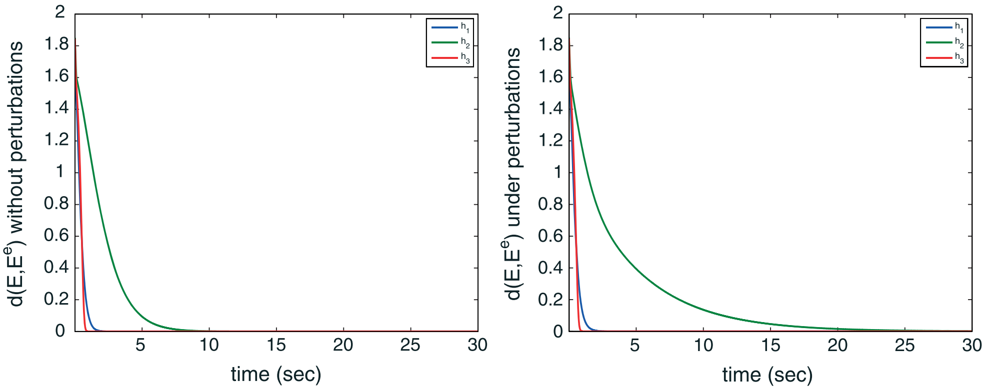
where $T_\sigma = 0.3$ and $l \in \{0, 1, 2, \ldots,\}$. It is easy to verify that $\mathcal{G}_{\sigma(t)}$ is JQSC.

We randomly choose the initial absolute attitudes of the system that satisfy $\boldsymbol{E}(0) \in \mathcal{R}$. Taking $k_2 = 0.5$, $b = 2$, we have $k_1 = \pi^2/4$, $k_3 \approx 10$ according to $h_1(\pi) = h_2(\pi) = h_3(\pi)$. Applying the angular velocity controller (22) to each agent, Figure 1(a) shows the time response curve of the distance from the trajectory $\boldsymbol{E}(t)$ to the desired equilibrium $\boldsymbol{E}^e$, and Figure 1(b) depicts the evolution of the angles of error attitudes $E_{12}, E_{13}, E_{14}, E_{15}$ with the potential function h_1.



(a) $d_{\mathcal{D}}(\boldsymbol{E}(t), \boldsymbol{E}^e)$ (b) Angles of error attitdues with h_1 (c) Angular velocities with h_1

Figure 1. Profiles of $d_{\mathcal{D}}(\boldsymbol{E}, \boldsymbol{E}^e)$, angles of error attitudes, and Euclidean norm of angular velocities with h_1. [Colour figure can be viewed at wileyonlinelibrary.com]



(a) $d_{\mathcal{D}}(\boldsymbol{E}(t), \boldsymbol{E}^e)$ without perturbations (b) $d_{\mathcal{D}}(\boldsymbol{E}(t), \boldsymbol{E}^e)$ under perturbations

Figure 2. Profile of $d_{\mathcal{D}}(\boldsymbol{E}, \boldsymbol{E}^e)$ with and without perturbations. [Colour figure can be viewed at wileyonlinelibrary.com]

The Euclidean norm of angular velocities is showed in Figure 1(c). These figures demonstrate that the desired attitude formation is achieved eventually as proved in Theorem 4.6.

Next, we suppose that the inter-agent graph $\mathcal{G}$ is a directed ring and consider the performances of these controllers under certain perturbations. Because $h_1''(0) = \pi^2/4$, $h_2''(0) = 1$, $h_3''(0) = 20$ are all strictly positive, the equilibrium point $\boldsymbol{E}^e$ is exponentially stable under the controller (26) with any of these three potential functions according to Theorem 5.1. Recalling Lemma 9.1 in [29], this exponential stability should be robust with respect to some vanishing perturbations with a small relative magnitude with respect to $d_{\mathcal{D}}(\boldsymbol{E}(t), \boldsymbol{E}^e)$. As that in [30], we add a linear additive perturbation $\Delta(t) = \gamma_p d_{\mathcal{D}}(\boldsymbol{E}(t), \boldsymbol{E}^e)$ to the angular velocity of each agent and choose γ_p at random between $[-0.1, 0.1]$. The evolution of the nominal system is depicted in Figure 2(a), while Figure 2(b) shows one of the perturbed cases. We can find the stability property still holds although the convergence rate is different, which verifies the robustness with respect to this special class of perturbations.

To illustrate the global behavior the formation control (26), we consider an undirected ring and take 1000 initial rotations $\boldsymbol{R}(0)$ that chosen uniformly at random in the whole state space $SO(3)^5$ and then conduct simulations by applying the control law (26) with (34). It turns out that there are 87, 443, and 0 number of failures to reach the desired formation with the respective functions h_1, h_2, and h_3. Figure 3(a) shows the curve of $d_{\mathcal{D}}(\boldsymbol{E}(t), \boldsymbol{E}^e)$ in one of the simulations that both the usage of h_1 and h_2 fail to reach the desired formation because the distance $d_{\mathcal{D}}(\boldsymbol{E}(t), \boldsymbol{E}^e)$ does not approach zero. In fact, from Figures 3(b) and 4, we can find the Euclidean norm of angular velocities of each agent under different potential functions vanishes as time goes to infinity, which implies the multi-agent system will converge to some steady state. While using the geodesic dependent

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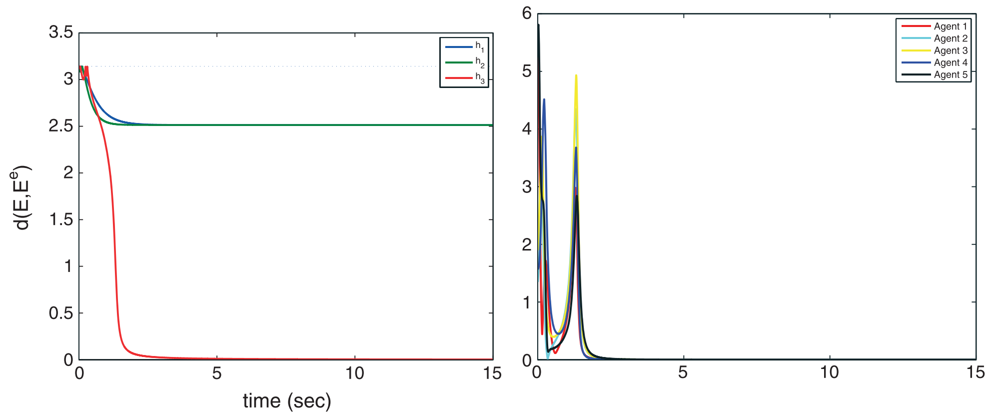
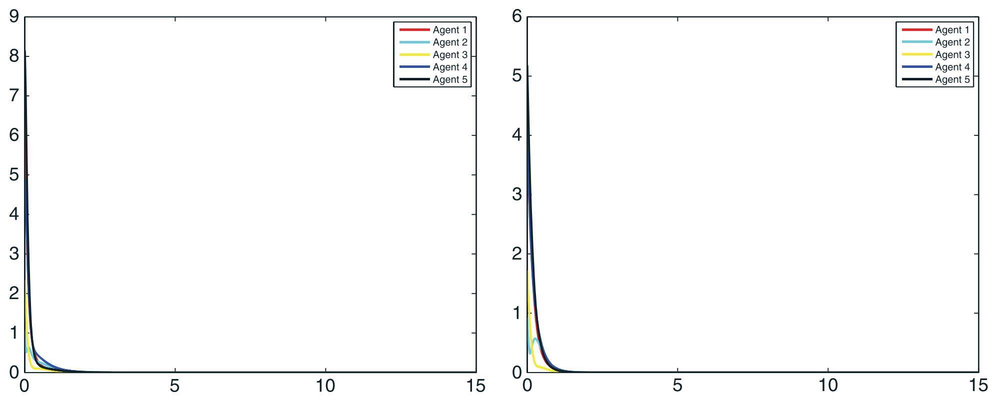


(a) $d_{\mathcal{D}}(\boldsymbol{E}(t), \boldsymbol{E}^e)$ with the undirected ring graph (b) Norm of angular velocities under h_3

Figure 3. Profile of $d_{\mathcal{D}}(\boldsymbol{E}, \boldsymbol{E}^e)$ when h_1 and h_2 fail and Euclidean norm of angular velocities with h_3. [Colour figure can be viewed at wileyonlinelibrary.com]



(a) Norm of angular velocities with h_1 (b) Norm of angular velocities with h_2

Figure 4. Euclidean norm of angular velocities with the potential functions h_1 and h_2. [Colour figure can be viewed at wileyonlinelibrary.com]

functions h_1 and h_2 might make the trajectories of the closed-loop system get trapped in some undesired equilibria, using the function (33) in Theorem 5.6 is able to deviate the trajectories from those undesired equilibria and achieve the attitude formation as depicted in Figure 3(a).

7. CONCLUSIONS

In this paper, we discussed the relative attitude formation control problem of multi-agent systems. We proposed a family of distributed angular velocity control laws using relative attitude information. Then, with switching and JQSC interaction topologies, we showed that the desired relative attitude formation is achieved asymptotically under certain mild assumptions on the initial relative attitudes between agents. Moreover, we also gave several sufficient conditions for the desired formation to be achieved exponentially and almost globally. Certainly, there are still many problems to be done in attitude formation based on relative information, which is still under our investigation.

APPENDIX A: PROOF OF LEMMA 3.3

Because $\mathrm{tr}(E_{ij}) = \mathrm{tr}(E_{ji})$ by (18), we obtain $\theta_{ij} = \theta_{ji}$ using (7). When $\theta_{ij} < \pi$, on the basis of (6) and (8), we obtain $\boldsymbol{k}_{ij} \sin\theta_{ij} = -R_{ij}^d \boldsymbol{k}_{ji} \sin\theta_{ji} = -R_{ij} \boldsymbol{k}_{ji} \sin\theta_{ji}$, which implies (19) by Assumption 3.1.

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Because the rotations $R_{ij}^d E_{jk} R_{ji}^d$ and E_{jk} are similar matrices, their angles are same by (7). Then from (18) and Lemma 2.3,

$$\cos\frac{\theta_{jk}}{2} = \left|\cos\frac{\theta_{ij}}{2}\cos\frac{\theta_{ik}}{2} + \sin\frac{\theta_{ij}}{2}\sin\frac{\theta_{ik}}{2}\boldsymbol{k}_{ij}^T\boldsymbol{k}_{ik}\right|.$$

If $\theta_{ij} + \theta_{ik} \leqslant \pi$, then $\theta_{jk} \leqslant \theta_{ij} + \theta_{ik}$ because of $\cos(\theta_{jk}/2) \geqslant \cos((\theta_{ij} + \theta_{ik})/2)$. Otherwise, $\theta_{ij} + \theta_{ik} > \pi \geqslant \theta_{jk}$. Hence, $\theta_{jk} \leqslant \theta_{ij} + \theta_{ik}$ always holds.

When $\theta_{jk} + \theta_{ij} + \theta_{ik} < 2\pi$, because of $\theta_{ij} + \theta_{jk} + \theta_{ik} \geqslant 2\theta_{ij}$, $\theta_{ij} < \pi$. Suppose $\cos(\theta_{jk}/2) = -\cos(\theta_{ij}/2)\cos(\theta_{ik}/2) - \sin(\theta_{ij}/2)\sin(\theta_{ik}/2)\boldsymbol{k}_{ij}^T\boldsymbol{k}_{ik}$. Then $\cos(\theta_{jk}/2) \leqslant -\cos(\theta_{ij}/2 + \theta_{ik}/2)$, which contradicts $\theta_{jk} + \theta_{ij} + \theta_{ik} < 2\pi$. Therefore,

$$\cos\frac{\theta_{jk}}{2} = \cos\frac{\theta_{ij}}{2}\cos\frac{\theta_{ik}}{2} + \sin\frac{\theta_{ij}}{2}\sin\frac{\theta_{ik}}{2}\boldsymbol{k}_{ij}^T\boldsymbol{k}_{ik}.$$

Because $\theta_{jk} \leqslant \theta_{ij} < \pi$, the previous equation implies $\sin(\theta_{ij}/2)\sin(\theta_{ik}/2)\boldsymbol{k}_{ij}^T\boldsymbol{k}_{ik} \geqslant 0$, and therefore, $h'(\theta_{ij})h'(\theta_{ik})\boldsymbol{k}_{ij}^T\boldsymbol{k}_{ik} \geqslant 0$ by Assumption 3.1. At the same time, $h'(\theta_{ij})h'(\theta_{ik})\boldsymbol{k}_{ij}^T\boldsymbol{k}_{ik} = 0$ if and only if $\cos(\theta_{jk}/2) = \cos(\theta_{ij}/2)\cos(\theta_{ik}/2)$, which is equivalent to $E_{ik} = I_3$ due to $\theta_{jk} \leqslant \theta_{ij}$. The other inequality in (b) can be verified similarly.

APPENDIX B: PROOFS OF LEMMAS 4.3–4.5

Proof of Lemma 4.3

Let $\tau_{a_0} = t'$ and $\{\tau_{a_l}\}_{l=1}^{\infty}$ be the (increasing) switching time sequence after τ_{a_0}. Then $\sigma(t)$ is constant on $[\tau_{a_l}, \tau_{a_{l+1}})$ for any $l \in \{0, 1, 2, \ldots\}$.

We first show that $(i, j) \notin \mathcal{K}_\alpha(t)$ for any $t \in [\tau_{a_0}, \tau_{a_1})$. Suppose this is not true, because of $(i, j) \notin \mathcal{K}_\alpha(\tau_{a_0})$, there is an $s \in (\tau_{a_0}, \tau_{a_1})$ such that $V_{ij}(\boldsymbol{E}(s)) = \alpha$. Then $(i, j) \in \mathcal{K}_\alpha(s)$ and $(i, j) \in \mathcal{I}(\boldsymbol{E}(s))$. From Lemma 4.1, there are two cases for $D^+V_{ij}(\boldsymbol{E}(s)) = \dot{V}_{ij}(\boldsymbol{E}(s))$, and we discuss them in the following respectively.

Case 1: $\dot{V}_{ij}(\boldsymbol{E}(s)) < 0$. Because $\dot{V}_{ij}(\boldsymbol{E}(\cdot))$ is continuous on (τ_{a_0}, τ_{a_1}), there is a $t'' \in (\tau_{a_0}, s)$ such that $\dot{V}_{ij}(\boldsymbol{E}(\cdot)) < 0$ on $(t'', s]$. Hence, $V_{ij}(\boldsymbol{E}(s)) < V_{ij}(\boldsymbol{E}(t'')) \leqslant \alpha$, which contradicts $(i, j) \in \mathcal{K}_\alpha(s)$.

Case 2: $\dot{V}_{ij}(\boldsymbol{E}(s)) = 0$. Denote the inter-agent graph on the time interval $[\tau_{a_0}, \tau_{a_1})$ as $\mathcal{G}^1$. Then at least one of agents i and j has a neighbor in $\mathcal{G}^1$; otherwise, $\dot{V}_{ij}(\boldsymbol{E}(\cdot)) \equiv 0$ on $[\tau_{a_0}, \tau_{a_1})$ by (23), which contradicts the fact that $(i, j) \notin \mathcal{K}_\alpha(\tau_{a_0})$ but $(i, j) \in \mathcal{K}_\alpha(s)$. Without loss of generality, suppose the neighbor set of agent i in $\mathcal{G}^1$ is nonempty. Let $\mathcal{U} = \{i\} \cup \{k \mid k \text{ is connected to } i \text{ in } \mathcal{G}^1\}$, and denote $\mathcal{N}_k^1$ as the neighbor set of any node $k \in \mathcal{U}$ in $\mathcal{G}^1$. Because $\mathcal{N}_k^1 \subset \mathcal{U}$ for any $k \in \mathcal{U}$, the motions of the error attitudes of agent pairs in $\mathcal{U} \times \mathcal{U}$ during $[\tau_{a_0}, \tau_{a_1})$ only depend on the motions of agents in $\mathcal{U}$ and are governed by

$$\begin{aligned} \dot{E}_{k_1k_2} &= \widehat{\boldsymbol{\omega}}_{k_1k_2}E_{k_1k_2} - E_{k_1k_2}\widehat{\boldsymbol{\omega}}_{k_1k_2}^d(t), \quad \forall k_1, k_2 \in \mathcal{U}, \\ \boldsymbol{\omega}_k &= \boldsymbol{\omega}_k^d(t) + \sum_{k' \in \mathcal{N}_k^1} h'(\theta_{ij})\boldsymbol{k}_{kk'}, \quad \forall k \in \mathcal{U}. \end{aligned} \tag{35}$$

Because $V_{ij}(\boldsymbol{E}(\tau_{a_0})) < \alpha$ and $V_{ij}(\boldsymbol{E}(s)) < \alpha$, the state of the closed-loop system (35) at time τ_{a_0} is not an equilibrium point of (35). At time instance s, because $(i, j) \in \mathcal{I}(\boldsymbol{E}(s))$ and $\dot{V}_{ij}(\boldsymbol{E}(s)) = 0$, $E_{ip}(s) = I_3$ for any $p \in \mathcal{N}_i^1$ by Lemma 4.1. Then for any $p \in \mathcal{N}_i^1$, $(p, j) \in \mathcal{K}_\alpha(s)$ because $E_{jp}(s) = E_{ji}(s)R_{ji}^d(s)E_{ip}(s)R_{ij}^d(s) = E_{ji}(s)$. For any $p \in \mathcal{N}_i^1$, if $\dot{V}_{pj}(\boldsymbol{E}(s)) < 0$, this turns to case 1 and leads to a contradiction. Thus, $\dot{V}_{pj}(\boldsymbol{E}(s)) = 0$ for any $p \in \mathcal{N}_i^1$, implying $E_{pr}(s) = I_3$ for any $r \in \mathcal{N}_p^1$. Repeating these arguments and combining (18), we obtain $E_{k_1k_2}(s) = I_3$ for any $k_1, k_2 \in \mathcal{U}$, which implies that the state of the closed-loop system (35) at time s is an equilibrium of (35). Therefore, the closed-loop system (35) has converged to an equilibrium point in finite time, which contradicts the uniqueness of the solution of (35).

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Then for any $t > \tau_{a_1}$, $V_{ij}(\boldsymbol{E}(t)) < \alpha$ as long as the graph $\mathcal{G}_{\sigma(t)}$ is not switched. Because $V_{ij}(\boldsymbol{E}(t))$ is continuous in t, this implies $V_{ij}(\boldsymbol{E}(t)) < \alpha$ for any $t \in [\tau_{a_0}, \tau_{a_1}]$, that is, $(i,j) \notin \mathcal{K}_\alpha(t)$ for any $t \in [\tau_{a_0}, \tau_{a_1}]$.

Because $(i,j) \notin \mathcal{K}_\alpha(\tau_{a_1})$, applying similar arguments on the time interval $[\tau_{a_1}, \tau_{a_2})$ reaches the conclusion that $(i,j) \notin \mathcal{K}_\alpha(t)$ for any $t \in [\tau_{a_1}, \tau_{a_2}]$. Repeating the previous arguments on the remaining time interval $[\tau_{a_2}, \tau_{a_3}), [\tau_{a_3}, \tau_{a_4}), \ldots$, respectively, we infer that $(i,j) \notin \mathcal{K}_\alpha(t)$ for any $t \geqslant t'$. □

Proof of Lemma 4.4

Because $\boldsymbol{E}(t_1) \neq \boldsymbol{E}^e$, $\alpha = V(\boldsymbol{E}(t_1)) > 0$.

Let $\mathcal{J}_\alpha(t) = \{i \in \mathcal{V} \mid \exists j \in \mathcal{V} \text{ such that } (i,j) \in \mathcal{K}_\alpha(t)\}$ be the set of agents in $\mathcal{K}_\alpha(t)$. Because $\mathcal{K}_\alpha(t_1) \neq \emptyset$, there are at least two agents in $\mathcal{J}_\alpha(t_1)$ and $\mathcal{J}_\alpha(t^2) \subset \mathcal{J}_\alpha(t^1)$ for any $t^2 \geqslant t^1 \geqslant t_1$ by Lemma 4.3. Let $\mathcal{J}_\alpha^1 = \mathcal{J}_\alpha(t_1)$ and $\mathcal{K}_\alpha^1 = \mathcal{K}_\alpha(t_1)$. We first show that at least one agent in $\mathcal{J}_\alpha^1$ will leave $\mathcal{J}_\alpha(t)$ at some time $t \in [t_1, t_1 + T)$. Suppose that this is not true, that is,

$$\mathcal{J}_\alpha(t) \equiv \mathcal{J}_\alpha^1, \quad \forall t \in [t_1, t_1 + T). \tag{36}$$

Then for any $t \in [t_1, t_1 + T)$, the following two statements hold

$$E_{ip}(t) \equiv I_3, \quad \forall i \in \mathcal{J}_\alpha^1, \quad \forall p \in \mathcal{N}_i(t), \tag{37}$$

$$\mathcal{K}_\alpha(t) \equiv \mathcal{K}_\alpha^1. \tag{38}$$

We obtain (37) because, for any $i \in \mathcal{J}_\alpha^1$, suppose there is a time $s \in [t_1, t_1 + T)$ such that $E_{ip}(s) \neq I_3$ for some $p \in \mathcal{N}_i(t)$, then take any $j \in \mathcal{V} \backslash \{i\}$,

(i) if $(i,j) \notin \mathcal{K}_\alpha(s)$, then $(i,j) \notin \mathcal{K}_\alpha(t)$ for any $t \geqslant s$ by Lemma 4.3.
(ii) if $(i,j) \in \mathcal{K}_\alpha(s)$, then $D^+V_{ij}(\boldsymbol{E}(s)) < 0$ by Lemma 4.1. Because $D^+V_{ij}(\boldsymbol{E}(\cdot))$ is continuous at non-switching time and right continuous at switching time, there is a $\delta > 0$ such that $D^+V_{ij}(\boldsymbol{E}(\cdot)) < 0$ on $[s, s+\delta)$. Hence, $V_{ij}(\boldsymbol{E}(t)) < V_{ij}(\boldsymbol{E}(s)) = \alpha$ for any $t \in (s, s+\delta)$. Then $(i,j) \notin \mathcal{K}_\alpha(t)$ for any $t > s$ by Lemma 4.3.

As a result, $i \notin \mathcal{J}_\alpha(t)$ for any $t > s$, which contradicts (36). Thus, (37) holds. Then from (23) and (37), for any $(i,j) \in \mathcal{K}_\alpha^1$, it holds that $D^+V_{ij}(\boldsymbol{E}(\cdot)) \equiv 0$ on $[t_1, t_1 + T)$, which implies (38).

Let $\mathcal{G}' = \mathcal{G}([t_1, t_1 + T))$. Then $\mathcal{G}'$ contains a root node that we designate as k_r. Choose any $(i,j) \in \mathcal{K}_\alpha^1$. We first consider the case that there is a directed path from j to i in $\mathcal{G}'$. Suppose there is a number of n_i nodes in the path. For convenience, we rename agent j as agent 1, the second agent in the path as agent 2, and so on. The last agent, namely, agent i, is consequently renamed as agent n_i, and we obtain $(n_i, 1) \in \mathcal{K}_\alpha(t)$ for any $t \in [t_1, t_1 + T)$ by (38). Because agent $n_i - 1$ is a neighbor of agent n_i at some time $s' \in [t_1, t_1 + T)$, $E_{n_i, n_i - 1}(s') = I_3$ by (37). Then $E_{1, n_i - 1}(s') = E_{1,n_i}(s') R_{1,n_i}^d(s') E_{n_i, n_i - 1}(s') R_{n_i,1}^d(s') = E_{1,n_i}(s')$, which implies $(n_i - 1, 1) \in \mathcal{K}_\alpha(s')$. Thus, $(n_i - 1, 1) \in \mathcal{K}_\alpha(t)$ for any $t \in [t_1, t_1 + T)$ by (38). Repeating the arguments, we obtain $(2,1) \in \mathcal{K}_\alpha(t)$ for any $t \in [t_1, t_1 + T)$, that is, $V_{21}(\boldsymbol{E}(t)) = h(\theta_{21}(t)) = \alpha$ for any $t \in [t_1, t_1 + T)$. Because agent 1 is a neighbor of agent 2 at some time $s'' \in [t_1, t_1 + T)$, $E_{21}(s'') = I_3$ by (37), which implies $\alpha = V_{21}(\boldsymbol{E}(s'')) = h(0) = 0$ and contradicts $\alpha > 0$. Similarly, for the case that there is a directed path from i to j in $\mathcal{G}'$, we also obtain $\alpha = 0$ and have a contradiction. Then we consider the case that neither i is connected to j nor j is connected to i in $\mathcal{G}'$. Because there is a directed path from k_r to i, we obtain $(j, k_r) \in \mathcal{K}_\alpha(t)$ for any $t \in [t_1, t_1 + T)$. Then because there is a directed path from k_r to j, this also leads to $\alpha = 0$ and contradicts $\alpha > 0$.

Therefore, at least one agent in $\mathcal{J}_\alpha(t_1)$ is not in the set $\mathcal{J}_\alpha(t_1 + T)$. Repeating the aforementioned arguments, we infer that there are at most two agents in the set $\mathcal{J}_\alpha(t_1 + (n-2)T)$. From the definition of $\mathcal{J}_\alpha$, it is impossible for the set only contains one agent. Hence, $\mathcal{J}_\alpha(t) = \emptyset$ for any $t \geqslant t_1 + (n-1)T$, which implies $\mathcal{K}_\alpha(t) = \emptyset$ for any $t \geqslant t_1 + (n-1)T$. □

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Proof of Lemma 4.5
Take any $\kappa \in (0, \tau_d)$, and let $t_2 = t_1 + (n-1)T + \kappa$. As t_1 varies on the interval $[0, \infty)$, the number of switching times of $\sigma(t)$ and the corresponding sequence of inter-agent graph on the time interval $(t_1 + \frac{\kappa}{2}, t_2 - \frac{\kappa}{2})$ vary.

Because of $\tau_d > 0$, there is an upper bound $M = \lfloor (n-1)T/\tau_d \rfloor + 1$ on the number of switching times during $(t_1 + \frac{\kappa}{2}, t_2 - \frac{\kappa}{2})$. Take any $c \in \{0, 1, 2, \ldots, M\}$, and suppose there are c times of switch during $(t_1 + \frac{\kappa}{2}, t_2 - \frac{\kappa}{2})$. For a fixed c, because the number of all possible inter-agent graphs is finite (equals m), the number of all possible sequences of inter-agent graph on $(t_1 + \frac{\kappa}{2}, t_2 - \frac{\kappa}{2})$ such that Lemma 4.4 hold is also finite, and we denote it as M_c. Then take any $s \in \{1, 2, \ldots, M_c\}$, and denote the corresponding inter-agent graph sequence on $(t_1 + \frac{\kappa}{2}, t_2 - \frac{\kappa}{2})$ as $\{\mathcal{G}_s^0, \mathcal{G}_s^1, \ldots, \mathcal{G}_s^c\}$. Let $\delta_{c,s} = \alpha - V(\boldsymbol{E}(t_2))$ and $\delta'_{c,s} = V(\boldsymbol{E}(t_1 + \frac{\kappa}{2})) - V(\boldsymbol{E}(t_2 - \frac{\kappa}{2}))$.

When $c = 0$, the inter-agent graph on $(t_1 + \frac{\kappa}{2}, t_2 - \frac{\kappa}{2})$ is a fixed graph, and $\delta'_{c,s} > 0$. Because $V(\boldsymbol{E}(t))$ is non-increasing, $\delta_{c,s} \geqslant \delta'_{c,s} > 0$.

When $c \neq 0$, let the (increasing) time sequence $\{\tau_{a_1}, \tau_{a_2}, \ldots, \tau_{a_c}\}$ be switching time instances on $(t_1 + \frac{\kappa}{2}, t_2 - \frac{\kappa}{2})$. Then the inter-agent graphs on $[t_1 + \frac{\kappa}{2}, \tau_{a_1}), [\tau_{a_1}, \tau_{a_2}), \ldots, [\tau_{a_c}, t_2 - \frac{\kappa}{2})$ are $\mathcal{G}_s^0, \mathcal{G}_s^1, \ldots, \mathcal{G}_s^c$, respectively. Let $\boldsymbol{\tau} = [\tau_{a_1}, \tau_{a_2}, \ldots, \tau_{a_c}]^T$, $D = \{\boldsymbol{\tau} \in \mathbb{R}^c : \tau_{a_1} > t_1 + \frac{\kappa}{2}, \tau_{a_c} < t_2 - \frac{\kappa}{2}, \tau_{a_{l+1}} - \tau_{a_l} \geqslant \tau_d, l = 1, 2, \ldots, c-1\}$, and $\bar{D}$ be the closure of D. From Lemma 4.2, $\delta'_{c,s} = \delta'_{c,s}(\boldsymbol{\tau})$ is continuous in $\boldsymbol{\tau}$ on $\bar{D}$, and from Lemma 4.4, $\delta'_{c,s}(\boldsymbol{\tau}) > 0$ for any $\boldsymbol{\tau} \in D$. Notice that when $\tau_{a_1} = t_1 + \frac{\kappa}{2}$ (or $\tau_{a_c} = t_2 - \frac{\kappa}{2}$), the inter-agent graph on the time interval $(t_1 + \frac{\kappa}{2}, t_2 - \frac{\kappa}{2})$ becomes $\{\mathcal{G}_s^1, \ldots, \mathcal{G}_s^c\}$ (or $\{\mathcal{G}_s^0, \mathcal{G}_s^1, \ldots, \mathcal{G}_s^{c-1}\}$), and $\delta'_{c,s}(\boldsymbol{\tau}) > 0$ might not hold. Because $\kappa < \tau_d$, $\sigma(t)$ switches at most once on the interval $[t_1, t_1 + \frac{\kappa}{2}]$ and $[t_2 - \frac{\kappa}{2}, t_2]$, respectively. The four cases of the switch are discussed as follows:

Case 1: no switch on both $[t_1, t_1 + \frac{\kappa}{2}]$ and $[t_2 - \frac{\kappa}{2}, t_2]$. Because of the inter-agent graph sequence on $[t_1, t_2)$ is still $\{\mathcal{G}_s^0, \mathcal{G}_s^1, \ldots, \mathcal{G}_s^c\}$, $\delta^1_{c,s}(\boldsymbol{\tau}) = V(\boldsymbol{E}(t_1)) - V(\boldsymbol{E}(t_2))$ is continuous in $\boldsymbol{\tau}$ on $\bar{D}$, and $\delta^1_{c,s}(\boldsymbol{\tau}) > 0$ for any $\boldsymbol{\tau} \in \bar{D}$. Because $\delta^1_{c,s}(\boldsymbol{\tau})$ attains a minimum on the compact set $\bar{D}$, $\delta_{c,s} \geqslant \min_{\boldsymbol{\tau} \in \bar{D}} \delta^1_{c,s}(\boldsymbol{\tau}) > 0$.

Case 2: one switch on $[t_1, t_1 + \frac{\kappa}{2}]$ and no switch on $[t_2 - \frac{\kappa}{2}, t_2]$. This implies $\tau_{a_1} \geqslant t_1 + \tau_d$. Let $D_2 = \{\boldsymbol{\tau} \in D : \tau_{a_1} \geqslant t_1 + \tau_d\}$. Because of the inter-agent graph sequence on $[t_1 + \frac{\kappa}{2}, t_2)$ is still $\{\mathcal{G}_s^0, \mathcal{G}_s^1, \ldots, \mathcal{G}_s^c\}$, $\delta^2_{c,s}(\boldsymbol{\tau}) = V(\boldsymbol{E}(t_1 + \frac{\kappa}{2})) - V(\boldsymbol{E}(t_2)) > 0$ is continuous in $\boldsymbol{\tau}$ on $\bar{D}_2$, and $\delta^2_{c,s}(\boldsymbol{\tau}) > 0$ for any $\boldsymbol{\tau} \in \bar{D}_2$. Because $\delta^2_{c,s}(\boldsymbol{\tau})$ attains a minimum on the compact set $\bar{D}_2$, $\delta_{c,s} \geqslant \min_{\boldsymbol{\tau} \in \bar{D}_2} \delta^2_{c,s}(\boldsymbol{\tau}) > 0$.

The analysis of the other two cases is similar to case 2 and is omitted here. And we reach the conclusion that $\delta_{c,s} > 0$.

Let $\delta_\alpha = \min_{c \in \{0,1,\ldots,M\}, s \in \{1,2,\ldots,M_c\}} \delta_{c,s}$. Then $\delta_\alpha > 0$ and $V(\boldsymbol{E}(t)) \leqslant V(\boldsymbol{E}(t_2)) \leqslant \alpha - \delta_\alpha$ for any $t \geqslant t_1 + (n-1)T + \tau_d$. Because of all possible forms of the switching signal $\sigma(t)$ on the time interval $[t_1, t_2]$ are considered, δ_α is independent of t_1. □

APPENDIX C: PROOFS OF LEMMAS 5.3 AND 5.4

Proof of Lemma 5.3
Define the potential function for the whole multi-agent system as

$$\varphi(\boldsymbol{R}) = \sum_{(i,j) \in \mathcal{E}} \varphi_{ij}(\boldsymbol{R}), \quad \text{where } \varphi_{ij}(\boldsymbol{R}) = h(\theta_{ij}).$$

Let $\nabla_{\boldsymbol{R}} \varphi = \left\{\nabla_{R_i} \varphi\right\}_{i \in \mathcal{V}}$. From Lemma 2.2, if $h'(\pi) \neq 0$ and $\boldsymbol{R} \in \Omega_{\mathcal{G},\pi}$, the gradient does not exist; otherwise,

$$\nabla_{R_i} \varphi = -R_i \sum_{j \in \mathcal{N}_i} h'(\theta_{ij}) \widehat{\boldsymbol{k}}_{ij}, \quad i = 1, 2, \ldots, n.$$

Hence, the trajectory $\boldsymbol{R}(t)$ of the closed-loop system (31) moves along the negative gradient of φ whenever $\nabla_{\boldsymbol{R}(t)} \varphi$ exists. Then the solutions of (31) starting from any point in $SO(3)^n$ either converge to the set on which the gradient of φ is zero or converge to the set on which the gradient of φ does not exist.

Next, we show that when $h'(\pi) > 0$, any $\boldsymbol{R} \in \Omega_{\mathcal{G},\pi}$ is not the local minima of the potential φ and therefore, cannot be stable. Take any $\bar{\boldsymbol{R}} = \{\bar{R}_i\}_{i\in\mathcal{V}} \in \Omega_{\mathcal{G},\pi}$, and let $\boldsymbol{R}(\varepsilon) = \{R_i(\varepsilon)\}_{i\in\mathcal{V}}$ be a curve in $SO(3)^n$ passing $\bar{\boldsymbol{R}}$ at $\varepsilon = 0$, where $R_i(\varepsilon) = \exp(\varepsilon\alpha_i\widehat{\boldsymbol{u}})\bar{R}_i$ and the parameters $\boldsymbol{u} \in \mathcal{S}^2$, $\alpha_i \in \mathbb{R}$. Then

$$E_{ij}(\boldsymbol{R}(\varepsilon)) = R_i^T(\varepsilon)R_j(\varepsilon)R_{ji}^d = \exp\left(\varepsilon(\alpha_j - \alpha_i)(\bar{R}_i^T\boldsymbol{u})^\wedge\right)E_{ij}(\bar{\boldsymbol{R}}).$$

Take any $(i,j) \in \mathcal{E}$. Denote $\theta_{ij}(\varepsilon)$ as the angle of $E_{ij}(\boldsymbol{R}(\varepsilon))$ and $(\bar{\theta}_{ij}, \bar{\boldsymbol{k}}_{ij})$ as the angle and axis of $E_{ij}(\bar{\boldsymbol{R}})$. If $\bar{\theta}_{ij} = \pi$, then from Lemma 2.3,

$$\cos\frac{\theta_{ij}(\varepsilon)}{2} = \sin\frac{|\varepsilon(\alpha_j - \alpha_i)|}{2}\left|\boldsymbol{u}^T\bar{R}_i\bar{\boldsymbol{k}}_{ij}\right|.$$

Computing the left and right derivative of $\varphi_{ij}(\boldsymbol{R}(\varepsilon))$ with respect to ε at $\varepsilon = 0$ yields

$$\dot{\varphi}_{ij}(\boldsymbol{R}(0^-)) = h'(\pi)|(\alpha_j - \alpha_i)\boldsymbol{u}^T\bar{R}_i\bar{\boldsymbol{k}}_{ij}|, \quad \dot{\varphi}_{ij}(\boldsymbol{R}(0^+)) = -h'(\pi)|(\alpha_j - \alpha_i)\boldsymbol{u}^T\bar{R}_i\bar{\boldsymbol{k}}_{ij}|.$$

Otherwise, $\bar{\theta}_{ij} < \pi$, in which case $\varphi_{ij}(\boldsymbol{R}(\varepsilon))$ is differentiable with respect to ε at $\varepsilon = 0$, and therefore, $\dot{\varphi}_{ij}(\boldsymbol{R}(0^-)) = \dot{\varphi}_{ij}(\boldsymbol{R}(0^+))$. Because $\bar{\boldsymbol{R}} \in \Omega_{\mathcal{G},\pi}$ and $h'(\pi) > 0$, there exist $\boldsymbol{u}$ and $\{\alpha_i\}_{i\in\mathcal{V}}$ such that $\dot{\varphi}(\boldsymbol{R}(0^-)) > \dot{\varphi}(\boldsymbol{R}(0^+))$, implying $\bar{\boldsymbol{R}}$ is not the local minima of φ. □

Proof of Lemma 5.4

Take any equilibrium $\boldsymbol{R}^* \in \mathcal{M}$ if $h'(\pi) = 0$ or any $\boldsymbol{R}^* \in \mathcal{M}\backslash\Omega_{\mathcal{G},\pi}$ if $h'(\pi) \neq 0$. Next, we compute the linearization of the system about $\boldsymbol{R}^*$. Notice that the linearization evolves in $\mathbb{R}^{3n}$ because of $SO(3)^n = 3n$.

Let $\varepsilon \in \mathbb{R}$ be a perturbation parameter, and suppose the initial state of (31) is a perturbation of $\boldsymbol{R}^*$ denoted by $\boldsymbol{R}^\varepsilon = \{R_i^*\exp(\varepsilon\alpha_i\widehat{\boldsymbol{u}}_i)\}_{i\in\mathcal{V}}$, where $\alpha_i > 0$ and $\boldsymbol{u}_i \in \mathcal{S}^2$. We use $\boldsymbol{R}(t;\varepsilon)$ to denote the trajectory of (31) starting form $\boldsymbol{R}^\varepsilon$ at $t = 0$. Then denote $\boldsymbol{\omega}_i(t;\varepsilon)$ as the angular velocity of agent i and $\theta_{ij}(t;\varepsilon)$ as the angle of the error attitude $E_{ij}(\boldsymbol{R}(t;\varepsilon)) = R_i^T(t;\varepsilon)R_j(t;\varepsilon)R_{ji}^d$. Notice that when $\varepsilon = 0$, $\boldsymbol{R}^\varepsilon = \boldsymbol{R}^*$. Hence, $R_i(\cdot;0) \equiv R_i^*$ and $\boldsymbol{\omega}_i(\cdot;0) \equiv \boldsymbol{0}_3$ on $[0,\infty)$ for any $i \in \mathcal{V}$.

Let $R_{i\varepsilon}(t) = (\partial R_i(t;\varepsilon)/\partial\varepsilon)|_{\varepsilon=0}$. Then $R_i^{*T}R_{i\varepsilon} \in so(3)$ because of $R_{i\varepsilon} \in T_{R_i(t;0)}SO(3)$. Let $\boldsymbol{x}_i = \left(R_i^{*T}R_{i\varepsilon}\right)^\vee$, and we denote the state of the linearization system as $\boldsymbol{x} = [\boldsymbol{x}_1^T, \ldots, \boldsymbol{x}_n^T] \in \mathbb{R}^{3n}$. From (31), the trajectory $R_i(t;\varepsilon)$ satisfies the following differential equation:

$$\dot{R}_i(t;\varepsilon) = R_i(t;\varepsilon)\sum_{j\in\mathcal{N}_i}\frac{h'(\theta_{ij}(t;\varepsilon))}{2\sin(\theta_{ij}(t;\varepsilon))}(R_i^T(t;\varepsilon)R_j(t;\varepsilon)R_{ji}^d - R_{ij}^dR_j^T(t;\varepsilon)R_i(t;\varepsilon)).$$

From the assumption of $\boldsymbol{R}^*$, both sides of the previous equation are differentiable with respect to ε at $\varepsilon = 0$, and we obtain

$$\begin{aligned} R_i^{*T}\dot{R}_{i\varepsilon} = &\sum_{j\in\mathcal{N}_i}\left(h''(\theta_{ij}^*) - h'(\theta_{ij}^*)\cot\theta_{ij}^*\right)\theta_{ij\varepsilon}\widehat{\boldsymbol{k}}_{ij}^* \\ &+ \sum_{j\in\mathcal{N}_i}\frac{h'(\theta_{ij}^*)}{2\sin\theta_{ij}^*}\left(E_{ij}^*(R_{ij}^d\boldsymbol{x}_j)^\wedge + (R_{ij}^d\boldsymbol{x}_j)^\wedge E_{ij}^{*T} - \widehat{\boldsymbol{x}}_iE_{ij}^* - E_{ij}^{*T}\widehat{\boldsymbol{x}}_i\right). \end{aligned}$$

Let $\theta_{ij\varepsilon}(t) = \left(\partial\theta_{ij}(t;\varepsilon)/\partial\varepsilon\right)\big|_{\varepsilon=0}$. From $\theta_{ij}(t;\varepsilon) = \arccos\left((\mathrm{tr}(R_i^T(t;\varepsilon)R_j^T(t;\varepsilon)R_{ji}^d) - 1)/2\right)$ and the identity (4), we obtain $\theta_{ij\varepsilon} = \boldsymbol{k}_{ij}^{*T}(R_{ij}^d\boldsymbol{x}_j - \boldsymbol{x}_i)$. Then combining the identities (3) and (9) derives

$$\dot{\boldsymbol{x}}_i = \sum_{j\in\mathcal{N}_i}\left[H(E_{ij}(\boldsymbol{R}^*))R_{ij}^d\boldsymbol{x}_j - H(E_{ij}(\boldsymbol{R}^*))^T\boldsymbol{x}_i\right], \quad i = 1, 2, \ldots, n.$$

Therefore, the linearization system about $\boldsymbol{R}^*$ is $\dot{\boldsymbol{x}} = J(\boldsymbol{R}^*)\boldsymbol{x}$. □

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